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$\mathcal{M}$-theory on $Spin(7)$ Manifolds, Fluxes and 3D, $\mathcal{N}$=1 Supergravity

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ABSTRACT

We calculate the most general causal $\mathcal{N} = 1$ three-dimensional, gauge invariant action coupled to matter in superspace and derive its component form using Ectoplasmic integration theory. One example of such an action can be obtained by compactifying $\mathcal{M}$-theory on a $Spin(7)$ holonomy manifold taking non-vanishing fluxes into account. We show that the resulting three-dimensional theory is in agreement with the more general construction. The scalar potential resulting from Kaluza-Klein compactification stabilizes all the moduli fields describing deformations of the metric except for the radial modulus. This potential can be written in terms of the superpotential previously discussed in the literature.

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1 Introduction

Component formulations of supergravity in various dimensions with extended supersymmetry have been known for a long time. In general, the extended supergravities can be obtained by dimensional reduction and truncation of higher dimensional supergravities. For example, a four-dimensional supergravity with $\mathcal{N}=1$ supersymmetry leads to a three-dimensional supergravity with $\mathcal{N}=2$ supersymmetry after compactification. For this reason the component form of three-dimensional $\mathcal{N}=2$ supergravity is known. Although there has been much activity in three dimensions [1, 2, 3, 4, 5, 6, 7, 8, 9], there is no general *off-shell* component or superspace formulation of three-dimensional $\mathcal{N}=1$ supergravity in the literature. There are, however, on-shell realisations with $\mathcal{N}\geq 1$ given in [10, 11].[2] The $\mathcal{N}=1$ theory cannot be obtained by dimensional reduction from a four-dimensional theory and requires a formal analysis. One of the goals of this paper is to derive the most general off-shell three-dimensional $\mathcal{N}=1$ supergravity action coupled to an arbitrary number of scalars and $U(1)$ gauge fields.

Although the off-shell formulation of $\mathcal{N}=1$ three-dimensional supergravity has been around since 1979 [12], there has been little work done on understanding this theory with the same precision and detail of the minimal supergravity in four dimensions. The spectrum of the $\mathcal{N}=1$ three-dimensional supergravity theory consists of a dreibein, a Majorana gravitino and a single real auxiliary scalar field. Since our formal analysis yields an off-shell formulation, we can freely add distinct super invariants to the action. The resulting theory corresponds to a non-linear sigma model and copies of $U(1)$ gauge theories coupled to supergravity. We will present the complete superspace formulation in the hope that the presentation will familiarize the reader with the techniques required to reach our goals.

Three-dimensional supergravity theories can be obtained from compactifications of $\mathcal{M}$-theory with non-vanishing fluxes.[3] Such theories were first considered in [15, 16], and later on generalized to the Type IIB theory in [17].

In order to compactify while preserving supersymmetry, we must consider internal manifolds that admit covariantly constant spinors. Once the background metric is chosen, the shape and size of the internal manifold can still be deformed, which leads to scalar fields in the low-energy effective supergravity theory, the so called moduli fields. If the compactified theory contains no scalar potential, the moduli fields can take any possible values and the theory loses predictive power. However, it was realized in [17, 18, 19, 20, 21, 22] that for string theory and $\mathcal{M}$-theory compactifications with non-vanishing fluxes a scalar potential emerges, which stabilizes many of the moduli fields. Therefore, predictions for the coupling constants can be made. In order to connect the compactified $\mathcal{M}$-theory with the superspace formulation it is necessary to integrate out the auxiliary fields in the latter theory by using

[2] We thank Michael Haack for bringing these references to our attention.

[3] Some early references about warped compactifications of string theory are [13, 14]

the algebraic equations of motion. This process shows how the scalar potential is naturally written in terms of the superpotential.

For various reasons three-dimensional compactifications of $\mathcal{M}$-theory with minimal supersymmetry are specially interesting. First, it has been suggested that compactifications to three dimensions with $\mathcal{N} = 1$ supersymmetry could naturally explain a small cosmological constant in four-dimensions [23, 24].

Second, recall that it was shown in [25, 26] that some particular type of compactifications of $\mathcal{M}$-theory to three dimensions with $\mathcal{N} = 2$ supersymmetry are the supergravity dual of the four-dimensional confining gauge theory found in [27]. A duality of this type is of practical interest since calculations in the strongly coupled gauge theory can be performed in the dual weakly coupled supergravity theory in the spirit of the AdS/CFT correspondence found in [28]. From the point of view of supersymmetry, 3D, $\mathcal{N} = 1$ gauge theories are similar to 4D, $\mathcal{N} = 0$ theories. So insight into 4D, $\mathcal{N} = 0$ gauge theories could be gained from studying the supergravity duals of 3D, $\mathcal{N} = 1$ theories. This is a developing area and a complete discussion is beyond the scope of this paper. Instead, we refer the interested reader to [15, 16, 17, 25, 26, 29, 30, 27, 31].

In order to compactify $\mathcal{M}$-theory to three dimensions, keeping only the minimal supersymmetry, we require an eight-dimensional internal manifold admitting a single covariantly constant spinor. In the case of a compact Riemannian internal space, this leads us uniquely to manifolds with $Spin(7)$ holonomy. Early papers about compactification of $\mathcal{M}$-theory on manifolds with exceptional holonomy are e.g. [32, 33]. The constraints on the fluxes following from supersymmetry for compactifications of $\mathcal{M}$-theory on $Spin(7)$ holonomy manifolds with background fields were first derived in [34]. It was later shown in [35] and [36], that these constraints can be derived from a superpotential, whose explicit form is in accordance with the conjecture made in [37]. Several interesting examples and aspects of these compactifications have been discussed in the literature (see e.g. [38, 39] and references therein). In this paper, we shall calculate the Kaluza-Klein compactification of $\mathcal{M}$-theory on a $Spin(7)$ holonomy manifold with non-vanishing fluxes. Our calculation is similar to that of [20] which was done in the context of $\mathcal{M}$-theory compactifications on conformally Calabi-Yau four-folds. We will see that the resulting scalar potential leads to the stabilization of all the moduli fields corresponding to deformations of the internal manifold except the radial modulus. This scalar potential can be expressed in terms of the superpotential which has appeared previously in the literature [35, 36, 37].

This paper is organized as follows: Section 2, is devoted to the introduction of the geometry and dynamics of three-dimensional $\mathcal{N} = 1$ supergravity coupled to matter. In section 2.1, we present the algebra of supercovariant derivatives which describes the superspace geometry. We then discuss Ectoplasmic integration, the technique used to calculate the density projector, which is required to integrate over curved supermanifolds. In section 2.2, we solve the Bianchi identities for a super three-form subject to the given algebra required

for Ectoplasmic integration. In section 2.3, we detail the use of Ectoplasm to calculate the density projector. In section 2.4, we complete the supergravity analysis by first deriving the component fields and then calculating the component action. We end the analysis by giving the supersymmetry transformations for the component fields and putting the component action on shell i.e. we remove the auxiliary fields by their algebraic equations of motion. Section 3 is devoted to the compactification of $\mathcal{M}$-theory on a $Spin(7)$ holonomy manifold. We start in section 3.1 by compacifying without background fluxes. In section 3.2, we take the background fluxes into account and derive the complete form of the bosonic part of the action. We show that this action is a special case of our findings in the general construction of section 2. In section 4, we give a summary of our results and comment on the physics implied by the explicit form of the scalar potential. We conclude this section with some open questions and directions for future investigations suggested by our findings. Finally, details related to our calculations are contained in the appendices. Appendix A contains our notations and conventions. Appendix B describes the derivation of the three-dimensional Fierz identities. In appendix C, we derive the closure of the three-dimensional super covariant derivative algebra. In appendix D, we provide relevant aspects related to manifolds with $Spin(7)$ holonomy. In appendix E, we perform the Kaluza-Klein compactification of the eleven-dimensional Einstein-Hilbert term on a $Spin(7)$ holonomy manifold.

2 Minimal $3D$ Supergravity Coupled to Matter

Using the Ectoplasmic Integration theorem we derive the component action for the general form of supergravity coupled to matter. The matter sector includes $U(1)$ gauge fields and a non-linear sigma model.

2.1 Supergeometry

Calculating component actions from manifestly supersymmetric supergravity descriptions is a complicated process. Knowing the supergravity density projector simplifies this process. The density projector arises from the following observation. Every supergravity theory that is known to possess an off-shell formulation can be be shown to obey an equation of the form:

$$\int d^D x \, d^{\mathcal{N}}\theta \; \mathrm{E}^{-1}\mathcal{L} \;=\; \int d^D x \; \mathrm{e}^{-1}(\mathcal{D}^{\mathcal{N}}\mathcal{L}|)\,, \tag{2.1.1}$$

for a superspace with space-time dimension D, and fermionic dimension $\mathcal{N}$. E^{-1} is the super determinant of the super frame fields $\mathrm{E}_A{}^M$, $\mathcal{D}^{\mathcal{N}}$ is a differential operator called the supergravity density projector, and the symbol $|$ denotes taking the anti-commuting coordinate to zero. This relation has been dubbed the Ectoplasmic Integration Theorem and

shows us that knowing the form of the density projector allows us to evaluate the component structure of any Lagrangian just by evaluating $(\mathcal{D}^{\mathcal{N}}\mathcal{L}|)$. Thus, the problem of finding components for supergravity is relegated to computing the density projector.

Two well defined methods for calculating the density projector exist in the literature. The first method is based on super p-forms and the Ethereal Conjecture. This conjecture states that in all supergravity theories, the topology of the superspace is determined *solely* by its bosonic submanifold. The second method is called the ectoplasmic normal coordinate expansion [40, 41], and explicitly calculates the density projector. The normal coordinate expansion provides a proof of the ectoplasmic integration theorem. Both of these techniques rely heavily on the algebra of superspace supergravity covariant derivatives. The covariant derivative algebra for three dimensional supergravity was first given in [12]. In this paper, we have modified the original algebra by coupling it to n $U(1)$, gauge fields:[4]

$$\begin{aligned}
[\nabla_\alpha\ ,\ \nabla_\beta\} &= (\gamma^c)_{\alpha\beta}\,\nabla_c\ -\ (\gamma^c)_{\alpha\beta}R\,\mathcal{M}_c\,,\\
[\nabla_\alpha\ ,\ \nabla_b\} &= \ +\tfrac{1}{2}(\gamma_b)_\alpha{}^\delta R\nabla_\delta\ +\ [-2(\gamma_b)_\alpha{}^\delta\Sigma_\delta{}^d\ -\ \tfrac{2}{3}(\gamma_b\gamma^d)_\alpha{}^\epsilon(\nabla_\epsilon R)]\mathcal{M}_d\\
&\quad +\ (\nabla_\alpha R)\mathcal{M}_b + \tfrac{1}{3}(\gamma_b)_\alpha{}^\beta W^I_\beta t_I\,,\\
[\nabla_a\ ,\ \nabla_b\} &= +2\epsilon_{abc}[\ -\Sigma^{\alpha c} - \tfrac{1}{3}(\gamma^c)^{\alpha\beta}(\nabla_\beta R)\ ]\nabla_\alpha\\
&\quad +\epsilon_{abc}[\ \widehat{\mathcal{R}}^{cd}\ +\ \tfrac{2}{3}\eta^{cd}(-2\nabla^2 R\ -\ \tfrac{3}{2}R^2)\ ]\mathcal{M}_d\\
&\quad +\tfrac{1}{3}\epsilon_{abc}(\gamma^c)_\beta{}^\delta\nabla^\beta W^I_\delta t_I\,,
\end{aligned} \tag{2.1.2}$$

where

$$\begin{aligned}
\widehat{\mathcal{R}}^{ab} - \widehat{\mathcal{R}}^{ba} = \eta_{ab}\widehat{\mathcal{R}}^{ab} &= (\gamma_d)^{\alpha\beta}\Sigma_\beta{}^d = 0\,,\\
\nabla_\alpha\Sigma_\beta^{\ f} &= -\tfrac{1}{4}(\gamma^e)_{\alpha\beta}\widehat{\mathcal{R}}_e{}^f + \tfrac{1}{6}[C_{\alpha\beta}\eta^{fd} + \tfrac{1}{2}\epsilon^{fde}(\gamma_e)_{\alpha\beta}]\nabla_d R\,,\\
\nabla^\delta W^I_\delta &= 0\,.
\end{aligned} \tag{2.1.3}$$

The superfields R, $\Sigma_\alpha{}^b$ and $\widehat{\mathcal{R}}^{ab}$ are the supergravity field strengths, and W^I_α are the $U(1)$ super Yang-Mills fields strengths. t_I are the $U(1)$ generators with $I = 1\ldots n$. $\mathcal{M}_a$ is the 3D Lorentz generator. Our convention for the action of $\mathcal{M}_a$ is given in appendix A.1. An explicit verification of the algebra (2.1.2) is performed in appendix C, where it is shown that the algebra closes off-shell.

In this paper, we choose to use the method of ectoplasmic integration. The following three sections outline the implementation of this procedure.

2.2 Closed Irreducible 3D, $\mathcal{N} = 1$ Super 3-forms

Indices of topological significance in a D-dimensional space-time manifold can be calculated from the integral of closed but not exact D-forms. The Ethereal Conjecture suggests

[4] We do not consider non-abelian gauged supergravity because the compactifications of $\mathcal{M}$-theory on $Spin(7)$ manifolds that we consider lead to abelian gauged supergravities.

that this reasoning should hold for superspace. Thus, in order to use the Ethereal Conjecture [42], we must first have the description of a super 3-form field strength. In this section, we derive the super 3-form associated with the covariant derivative algebra (2.1.2).

We start with the general formulas for the super 2-form potential and super 3-form field strength. A super 2-form Γ_2 has the following gauge transformations:

$$\delta\Gamma_{AB} = \nabla_{[A}K_{B)} - \tfrac{1}{2}T_{[AB)}{}^{E}K_{E} \quad , \tag{2.2.1}$$

which expresses the fact that the gauge variation of the super 2-form is the super-exterior derivative of a super 1-form K_1. The field strength G_3 is the super exterior derivative of Γ_2:

$$G_{ABC} = \tfrac{1}{2}\nabla_{[A}\Gamma_{BC)} - \tfrac{1}{2}T_{[AB|}{}^{E}\Gamma_{E|C)} \quad . \tag{2.2.2}$$

We have a few comments about the notation in these expressions. First, upper case roman indices are super vector indices which take values over both the spinor and vector indices. Also, letters from the beginning(middle) of the alphabet refer to flat(curved) indices. Finally, the symmetrization symbol [) is a graded symmetrization. A point worth noting here is the that the superspace torsion appears explicitly in these equations. This means that the super-form is intimately related to the type of supergravity that we are using. The appearance of the torsion in these expressions is *not* peculiar to supersymmetry. Whenever forms are referred to using a non-holonomic basis this phenomenon occurs.

A super-form is a highly reducible representation of supersymmetry. Therefore, we must impose certain constraints on the field strength to make it an irreducible representation of supersymmetry. In general, there are many types of constraints that we can set. Different constraints have specific consequences. A conventional constraint implies that one piece of the potential is related to another. In this case if we set the conventional constraint:

$$G_{\alpha\beta\gamma} = \tfrac{1}{2}\nabla_{(\alpha}\Gamma_{\beta\gamma)} - \tfrac{1}{2}(\gamma^{c})_{(\alpha\beta|}\Gamma_{c|\gamma)} = 0 \quad , \tag{2.2.3}$$

we see that the potential $\Gamma_{c\alpha}$ is now related to the spinorial derivative of the potential $\Gamma_{\alpha\beta}$. This constraint eliminates six superfield degrees of freedom.

Since G_3 is the exterior derivative of a super 2-form it must be closed, i.e. its exterior derivative F_4 must vanish. This constitutes a set of Bianchi identities:

$$F_{ABCD} = \tfrac{1}{3!}\nabla_{[A}G_{BCD)} - \tfrac{1}{4}T_{[AB|}{}^{E}G_{E|CD)} = 0 \quad . \tag{2.2.4}$$

Once a constraint has been set, these Bianchi identities are no longer identities. In fact, the consistency of the Bianchi identities after a constraint has been imposed implies an entire set of constraints. By solving the Bianchi identities with respect to the conventional constraint, we can completely determine the irreducible super 3-form field strength. Since we have set $G_{\alpha\beta\gamma} = 0$, it is easiest to solve $F_{\alpha\beta\gamma\delta} = 0$ first:

$$F_{\alpha\beta\gamma\delta} \;=\; \tfrac{1}{6}\nabla_{(\alpha}G_{\beta\gamma\delta)} - \tfrac{1}{4}T_{(\alpha\beta|}{}^{E}G_{E|\gamma\delta)} = -\tfrac{1}{4}(\gamma^{e})_{(\alpha\beta|}G_{e|\gamma\delta)} \quad . \tag{2.2.5}$$

To solve this equation, we must write out the Lorentz irreducible parts of $G_{a\beta\gamma}$. We first convert the last two spinor indices on $G_{e\gamma\delta}$ to a vector index by contracting with the gamma matrix: $G_{e\gamma\delta} = (\gamma^f)_{\gamma\delta} G_{ef}$. Further, $G_{a\beta\gamma} = G_{\beta\gamma a}$ implies that G_{ab} is a symmetric tensor, so we make the following decomposition: $G_{ab} = \overline{G}_{ab} + \frac{1}{3}\eta_{ab} G^d{}_d$, where the bar on $\overline{G}_{ab}$ denotes tracelessness. With this decomposition, the Bianchi identity now reads:

$$F_{\alpha\beta\gamma\delta} = -\tfrac{1}{4}(\gamma^e)_{(\alpha\beta}(\gamma^f)_{\gamma\delta)}\overline{G}_{ef} = 0 \quad , \tag{2.2.6}$$

where the term containing $G^d{}_d$ vanishes exactly. The symmetric traceless part of this gamma matrix structure does not vanish, so we are forced to set $\overline{G}_{ab} = 0$. Thus, our conventional constraint implies the further constraint $G_{a\beta\gamma} = (\gamma_a)_{\beta\gamma} G$. The next Bianchi identity reads:

$$F_{\alpha\beta\gamma d} = \tfrac{1}{2}\nabla_{(\alpha} G_{\beta\gamma)d} - \tfrac{1}{3!}\nabla_d G_{(\alpha\beta\gamma)} - \tfrac{1}{2} T_{(\alpha\beta|}{}^E G_{E|\gamma)d} + \tfrac{1}{2} T_{d(\alpha|}{}^E G_{E|\beta\gamma)} \quad . \tag{2.2.7}$$

Using our newest constraint and substituting the torsions we have:

$$\begin{aligned} F_{\alpha\beta\gamma d} &= \tfrac{1}{2}(\gamma_d)_{(\beta\gamma}\nabla_{\alpha)} G + \tfrac{1}{2}(\gamma^e)_{(\alpha\beta|} G_{\gamma)ed} \\ &= \tfrac{1}{2}(\gamma_d)_{(\beta\gamma}\nabla_{\alpha)} G + \tfrac{1}{2}(\gamma^e)_{(\alpha\beta|}\epsilon_{ed}{}^a[(\gamma_a)_{\gamma)}{}^\delta G_\delta + \hat{G}_{\gamma)a}] \quad , \end{aligned} \tag{2.2.8}$$

here we have replaced the antisymmetric vector indices with a Levi-Cevita tensor via; $G_{\gamma ed} = \epsilon_{ed}{}^a G_{\gamma a}$, and further decomposed $G_{\gamma a}$ into spinor and gamma traceless parts; $G_{\gamma a} = (\gamma_a)_\gamma{}^\beta G_\beta + \hat{G}_{\gamma a}$, respectively. Contracting (2.2.8) with $\epsilon_{cbe}(\gamma^e)^{\alpha\beta}\delta_\sigma{}^\gamma$ implies $G_\sigma = \nabla_\sigma G$. Substituting this result back into (2.2.8) implies that $\hat{G}_\gamma{}^a = 0$. Thus, we have derived another constraint on the field strength:

$$G_{\alpha bc} = \epsilon_{bc}{}^a(\gamma_a)_\alpha{}^\sigma \nabla_\sigma G \quad . \tag{2.2.9}$$

The third bianchi identity will completely determine the super 3-form:

$$\begin{aligned} F_{\alpha\beta cd} &= \nabla_{(\alpha} G_{\beta)cd} + \nabla_{[c} G_{d]\alpha\beta} - T_{\alpha\beta}{}^E G_{Ecd} - T_{cd}{}^E G_{E\alpha\beta} - T_{(\alpha|[c|}{}^E G_{E|d]\beta)} \\ &= \epsilon_{cd}{}^e(\gamma_e)_{(\alpha}{}^\sigma\nabla_{\beta)}\nabla_\sigma G + (\gamma_{[d})_{\alpha\beta}\nabla_{c]} G - (\gamma^e)_{\alpha\beta} G_{ecd} + (\gamma_{[c}\gamma_{d]})_{\alpha\beta} R G \ . \end{aligned} \tag{2.2.10}$$

Note that $G_{\alpha b\gamma} = -(\gamma_b)_{\alpha\gamma} G$. Contracting with $(\gamma_b)^{\alpha\beta}$ yields the following equation for the vector 3-form:

$$G_{bcd} = 2\epsilon_{bcd}[\nabla^2 G + RG] \quad . \tag{2.2.11}$$

The final two bianchi identities are consistency checks and vanish identically.

$$\begin{aligned} F_{\alpha bcd} &= \tfrac{1}{3!}\nabla_\alpha G_{[bcd]} - \tfrac{1}{2}\nabla_{[b} G_{cd]\alpha} - \tfrac{1}{2} T_{\alpha[b|}{}^E G_{E|cd]} + \tfrac{1}{2} T_{[bc|}{}^E G_{E|d]\alpha} = 0 \\ F_{abcd} &= \tfrac{1}{3!}\nabla_{[a} G_{bcd]} - \tfrac{1}{4} T_{[ab|}{}^E G_{E|cd]} = 0 \end{aligned} \tag{2.2.12}$$

We have shown that the super 3-form field strength related to the supergravity covariant derivative algebra (2.1.2) is completely determined in terms of a scalar superfield G. In 3D, a scalar superfield is an irreducible representation of supersymmetry, and therefore the one conventional constraint was enough to completely reduce the super 3-form.

2.3 Ectoplasmic Integration

In order to use the Ethereal Conjecture, we must integrate a 3-form over the bosonic sub-manifold. The super 3-form derived in the previous section is:

$$
\begin{gathered}
G_{\alpha\beta\gamma} = 0 \quad , \\
G_{\alpha\beta c} = (\gamma_c)_{\alpha\beta} G \quad , \\
G_{\alpha bc} = \epsilon_{bcd} (\gamma^d)_\alpha{}^\sigma \nabla_\sigma G \quad , \\
G_{abc} = 2\epsilon_{abc} [\nabla^2 G + RG] \quad .
\end{gathered} \tag{2.3.1}
$$

The only problem with this super 3-form is that it has flat indices. We worked in the tangent space so that we could set supersymmetric constraints on the super 3-form. Now we require the curved super 3-form to find the generally covariant component 3-form. In general, the super 3-form with flat indices is related to the super 3-form with curved indices via:[5]

$$
\mathcal{G}_{MNO} = (-)^{[\frac{3}{2}]} \mathrm{E}_M{}^A \mathrm{E}_N{}^B \mathrm{E}_O{}^C G_{CBA} \quad . \tag{2.3.2}
$$

As it turns out, the component 3-form is the lowest component of the curved super 3-form $g_{mno} = \mathcal{G}_{mno}|$. Using the usual component definitions for the super frame fields; $\mathrm{E}_m{}^a| = e_m{}^a$, $\mathrm{E}_m{}^\alpha| = -\psi_m{}^\alpha$, we can write the lowest component of the vector 3-form part of (2.3.2):

$$
g_{mno} = -G_{onm}| - \tfrac{1}{2}\psi_{[m}{}^\alpha G_{no]\alpha}| - \tfrac{1}{2}\psi_{[m}{}^\alpha \psi_n{}^\beta G_{o]\alpha\beta}| + \psi_m{}^\alpha \psi_n{}^\beta \psi_o{}^\gamma G_{\alpha\beta\gamma}| \quad . \tag{2.3.3}
$$

Since this is a θ independent equation, we can convert all of the curved indices to flat ones using $\mathrm{e}_a{}^m$:

$$
\begin{aligned}
g_{abc} &= G_{abc}| - \tfrac{1}{2}\psi_{[a}{}^\alpha G_{bc]\alpha}| - \tfrac{1}{2}\psi_{[a}{}^\alpha \psi_b{}^\beta G_{c]\alpha\beta}| + \psi_a{}^\alpha \psi_b{}^\beta \psi_c{}^\gamma G_{\alpha\beta\gamma}| \\
&= 2\epsilon_{abc}[\nabla^2 G| + R|G|] - \tfrac{1}{2}\psi_{[a}{}^\alpha \epsilon_{bc]d}(\gamma^d)_\alpha{}^\sigma \nabla_\sigma G| - \tfrac{1}{2}\psi_{[a}{}^\alpha \psi_b{}^\beta (\gamma_{c]})_{\alpha\beta} G| \\
&= \Big\{ 2\epsilon_{abc}[\,\nabla^2 + R|\,] - \tfrac{1}{2}\psi_{[a}{}^\alpha \epsilon_{bc]d}(\gamma^d)_\alpha{}^\sigma \nabla_\sigma - \tfrac{1}{2}\psi_{[a}{}^\alpha \psi_b{}^\beta (\gamma_{c]})_{\alpha\beta} \Big\} G| \quad .
\end{aligned} \tag{2.3.4}
$$

We note in passing that this equation is of the form $\mathcal{D}^2 G|$. Since g_{abc} is part of a closed super 3-form, it is also closed in the ordinary sense. Thus, any volume 3-form $\omega^{abc} = \omega\, \epsilon^{abc}$ may be integrated against g_{abc} and will yield an index of the 3D theory if g_{abc} is not exact. We are led to define an index Δ by

$$
\Delta \;=\; \int \omega\, \epsilon^{abc}\, g_{abc} \quad . \tag{2.3.5}
$$

If we define $\frac{1}{6}\epsilon^{abc} g_{abc} = \mathcal{D}^2 G|$ we can read off the density projector:

$$
\mathcal{D}^2 = -2\nabla^2 + \psi_d{}^\alpha (\gamma^d)_\alpha{}^\sigma \nabla_\sigma - \tfrac{1}{2}\psi_a{}^\alpha \psi_b{}^\beta \epsilon^{abc} (\gamma_c)_{\alpha\beta} - 2R \quad . \tag{2.3.6}
$$

The Ethereal Conjecture asserts that for all superspace Lagrangians $\mathcal{L}$ the local integration theory for 3D, $\mathcal{N} = 1$ superspace supergravity takes the form:

$$
\int d^3x d^2\theta \mathrm{E}^{-1} \mathcal{L} = \int d^3\, \mathrm{e}^{-1} (\mathcal{D}^2 \mathcal{L}|) \quad . \tag{2.3.7}
$$

[5] We have used a different symbol for the curved super 3-form just to avoid any possible confusion.

2.4 Obtaining Component Formulations

We are interested in describing at the level of component fields the following general gauge invariant Lagrangian containing two derivatives for 3D, $\mathcal{N}=1$ gravity coupled to matter:

$$\begin{aligned}\mathcal{L} \;=\; & \kappa^{-2}K(\Phi)R \;+\; g^{-2}h(\Phi)_{IJ}W^{\alpha I}W^{J}_{\alpha} \;+\; g(\Phi)_{ij}\nabla^{\alpha}\Phi^{i}\nabla_{\alpha}\Phi^{j} \\ & +\; Q_{IJ}\Gamma^{I}_{\beta}W^{J\beta} \;+\; W(\Phi) \quad .\end{aligned} \tag{2.4.1}$$

This action encompasses all possible terms which can arise from the compactification of M-theory which we are considering. The first term is exactly 3D supergravity when $K(\Phi)=1$. The second term is the kinetic term for the gauge fields. The third term is the kinetic part of the sigma model for the scalar matter fields Φ^{i}. The fourth term represents the Chern-Simons term for the gauge fields. Finally, $W(\Phi)$ is the super potential.

In order to obtain the usual gravity fields we must know how to define the components of the various field strengths and curvatures. This is done in a similar manner as before when we determined the 3-form component field of the super 3-form. In this case, we go to a Wess-Zumino gauge to write all of the torsions, curvatures and field strengths at $\theta=0$:

$$\begin{aligned}T_{ab}{}^{\gamma}| \;&=\; t_{ab}{}^{\gamma} \;+\; \psi_{[a}{}^{\delta}T_{\delta b]}{}^{\gamma}| \;+\; \psi_{[a}{}^{\delta}\psi_{b]}{}^{\gamma}T_{\delta\gamma}{}^{\gamma}| \quad , \\ T_{ab}{}^{c}| \;&=\; t_{ab}{}^{c} \;+\; \psi_{[a}{}^{\delta}T_{\delta b]}{}^{c}| \;+\; \psi_{[a}{}^{\delta}\psi_{b]}{}^{\gamma}T_{\delta\gamma}{}^{c}| \quad , \\ R_{ab}{}^{c}| \;&=\; r_{ab}{}^{c} \;+\; \psi_{[a}{}^{\delta}R_{\delta b]}{}^{c}| \;+\; \psi_{[a}{}^{\delta}\psi_{b]}{}^{\gamma}R_{\delta\gamma}{}^{c}| \quad , \\ \mathcal{F}_{ab}{}^{I}| \;&=\; f_{ab}{}^{I} \;+\; \psi_{[a}{}^{\delta}\mathcal{F}_{\delta b]}{}^{I}| \;+\; \psi_{[a}{}^{\delta}\psi_{b]}{}^{\gamma}\mathcal{F}_{\delta\gamma}{}^{I}| \quad .\end{aligned} \tag{2.4.2}$$

The leading terms in each of these $t_{ab}{}^{\gamma}$, $t_{ab}^{\;c}$, $r_{ab}{}^{c}$ and $f_{ab}{}^{I}$ correspond, respectively, to the exterior derivatives of $\psi_{a}{}^{\gamma}$, $\mathrm{e}_{a}{}^{m}$, $\omega_{a}{}^{c}$ and $A_{a}{}^{I}$, using the bosonic truncation of the definition of the exterior derivative that appears in (2.2.1). By definition the super-covariantized curl of the gravitino is the lowest component of the torsion $T_{ab}{}^{\gamma}$. Substituting from (2.1.2) we have:

$$f_{ab}{}^{\gamma} := T_{ab}{}^{\gamma}| = -2\epsilon_{abc}\Big[\,\Sigma^{\gamma c}| + \tfrac{1}{3}(\gamma^{c})^{\gamma\beta}(\nabla_{\beta}R)|\,\Big] \quad . \tag{2.4.3}$$

This equation implies the following:

$$\nabla_{\alpha}R| = -\tfrac{1}{4}(\gamma_{d})_{\alpha\gamma}\epsilon^{abd}f_{ab}^{\gamma} \quad , \quad \Sigma^{\gamma d}| = \tfrac{1}{6}\Big[\epsilon^{abd}f_{ab}{}^{\gamma} - (\gamma_{b})_{\delta}{}^{\gamma}f^{bd\delta}\Big] \quad . \tag{2.4.4}$$

The lowest component of $\Sigma^{\delta d}$ is indeed gamma traceless. The other torsion yields information about the component torsion:

$$T_{ab}{}^{c}| = 0 = t_{ab}{}^{c} + (\gamma^{c})_{\alpha\beta}\psi_{[a}{}^{\alpha}\psi_{b]}{}^{\beta} \quad , \tag{2.4.5}$$

which can be solved in the usual manner to express the spin-connection in terms of the anholonomy and gravitino. The super curvature leads us to the component definition of the super covariantized curvature tensor:

$$R_{ab}{}^{c}| \;=\; r_{ab}{}^{c} + \psi_{[a}{}^{\delta}[-2(\gamma_{b]})_{\delta}{}^{\alpha}\Sigma_{\alpha}{}^{c}| - \tfrac{2}{3}(\gamma_{b]}\gamma^{c})_{\delta}{}^{\alpha}\nabla_{\alpha}R| + \nabla_{\delta}R|\delta_{b]}{}^{c}] - \psi_{[a}{}^{\delta}\psi_{b]}{}^{\gamma}(\gamma^{c})_{\delta\gamma}R|$$

$$= \quad \epsilon_{abd}[\widehat{\mathcal{R}}^{dc}| + \tfrac{2}{3}\eta^{cd}(-2\nabla^2 R| - \tfrac{3}{2}R^2|)] \quad . \tag{2.4.6}$$

Contracting this equation with $\epsilon^{ab}{}_c$ and using the component definitions (2.4.4) leads to the component definition:

$$\nabla^2 R| = -\tfrac{3}{4}R^2| - \tfrac{1}{4}\epsilon^{abc}\psi_a{}^\alpha\psi_b{}^\beta(\gamma_c)_{\alpha\beta}R| + \tfrac{1}{8}\epsilon^{abc}r_{abc} + \tfrac{1}{4}\psi^{a\beta}(\gamma^b)_\beta{}^\gamma f_{ab\gamma} \quad . \tag{2.4.7}$$

The super field strength satisfies:

$$\mathcal{F}_{ab}{}^I| = f_{ab}{}^I + \tfrac{1}{3}\psi_{[a}{}^\delta(\gamma_{b]})_\delta{}^\alpha W^I_\alpha| = \tfrac{1}{3}\epsilon_{abc}(\gamma^c)_\beta{}^\delta\nabla^\beta W^I_\delta| \quad ,$$

which implies:

$$\nabla_\alpha W^I_\beta| = -\tfrac{3}{4}\epsilon^{abc}(\gamma_c)_{\alpha\beta}f^I_{ab} - \tfrac{1}{2}\epsilon^{abc}(\gamma_c)_{\alpha\beta}(\gamma_b)_\delta{}^\gamma\psi_a{}^\delta W^I_\gamma| \quad . \tag{2.4.8}$$

From (2.1.3) we have $\nabla_\alpha\nabla^\beta W^I_\beta = 0$ so we can derive:

$$\nabla^2 W^I_\alpha| = \tfrac{1}{2}(\gamma^c)_\alpha{}^\beta\nabla_c W^I_\beta| - \tfrac{3}{4}R|W^I_\alpha| \quad . \tag{2.4.9}$$

We now have complete component definitions for R and W_α and enough of the components of $\Sigma^{a\beta}$ and $\hat{R}_{ab}$ to perform the ectoplasmic integration. Since the gauge potential Γ^I_α for the $U(1)$ fields appears in our Lagrangian we must also make component definitions for it. Γ^I_α has the gauge transformation:

$$\delta\Gamma^I_\alpha = \nabla_\alpha K^I \quad , \tag{2.4.10}$$

so we can choose the Wess-Zumino gauge:

$$\Gamma^I_\alpha| = \nabla^\alpha\Gamma^I_\alpha| = 0 \quad . \tag{2.4.11}$$

We are now in a position to derive the full component action. We introduce the following definitions for the component fields

$$\begin{gathered} R| = B \qquad W^I_\alpha| = \lambda^I_\alpha \\ \Phi^i| = \phi^i \qquad \nabla_\alpha\Phi^i| = \chi^i_\alpha \qquad \nabla^2\Phi^i| = F^i \\ \nabla_\alpha\Gamma^I_\beta| = \tfrac{1}{2}(\gamma^c)_{\alpha\beta}A^I_c \qquad\qquad \nabla^\beta\nabla_\beta\Gamma^I_\alpha| = \tfrac{2}{3}\lambda^I_\alpha \quad , \end{gathered} \tag{2.4.12}$$

in addition to the curl of the gravitino defined in (2.4.2). Using these component definitions the terms in the action become:

$$\begin{aligned} \tfrac{1}{\kappa^2}\int d^3x d^2\theta \mathrm{E}^{-1}K(\Phi)R \;&=\; \tfrac{1}{\kappa^2}\int d^3x\, \mathrm{e}^{-1}(\mathcal{D}^2 K(\Phi)R)| \\ =\; \tfrac{1}{\kappa^2}\int d^3x\, \mathrm{e}^{-1}\Big\{ &-2B\nabla^2 K| + \nabla_\alpha K|\Big[-\tfrac{1}{2}(\gamma_a)_\beta{}^\alpha\epsilon^{abc}f_{bc}{}^\beta - \psi_d{}^\beta(\gamma^d)_\beta{}^\alpha B\Big] \\ &+K|\Big[-\tfrac{1}{2}B^2 - \tfrac{1}{4}\epsilon^{abc}r_{abc} + \tfrac{1}{4}\psi_a{}^\beta\epsilon^{abc}f_{bc\beta}\Big]\;\Big\} \quad , \end{aligned} \tag{2.4.13}$$

$$\tfrac{1}{g^2}\int d^3x d^2\theta \mathrm{E}^{-1}h_{IJ}W^{\alpha I}W_\alpha{}^J = \tfrac{1}{g^2}\int d^3x\, \mathrm{e}^{-1}(\mathcal{D}^2 h_{IJ}W^{\alpha I}W_\alpha{}^J)|$$

$$
\begin{aligned}
= \tfrac{1}{g^2}\int d^3x\,\mathrm{e}^{-1}\Big\{ &+\nabla_\alpha h_{IJ}|\Big[-3\epsilon^{abc}(\gamma_c)^{\alpha\beta} f_{ab}{}^I\lambda_\beta{}^J + (\gamma^a)_\delta{}^\alpha \psi_a{}^\delta \lambda^{\beta I}\lambda_\beta{}^J\Big] \\
&+h_{IJ}|\Big[-2(\gamma^c)^{\alpha\beta}(\nabla_c\lambda_\alpha{}^J)\lambda_\beta{}^J - \tfrac12\psi^{a\alpha}\psi_{a\alpha}\lambda^{\beta I}\lambda_\beta{}^J + 3(\gamma_e)_\sigma{}^\rho f^{deI}\psi_d{}^\sigma\lambda_\rho{}^J \\
&+B\lambda^{\beta I}\lambda_\beta{}^J + \tfrac92 f^{abI}f_{ab}{}^J - \tfrac32\epsilon^{abc}\psi_c{}^\gamma\lambda_\gamma{}^J f_{ab}{}^I\Big] - 2\nabla^2 h_{IJ}|\lambda^{\beta I}\lambda_\beta{}^J\Big\} \quad , \quad (2.4.14)
\end{aligned}
$$

$$
\begin{aligned}
\int d^3x d^2\theta \mathrm{E}^{-1} g_{ij}\nabla^\alpha\Phi^i\nabla_\alpha\Phi^j &= \int d^3x\,\mathrm{e}^{-1}(\mathcal{D}^2 g_{ij}\nabla^\alpha\Phi^i\nabla_\alpha\Phi^j)| \\
= \int d^3x\,\mathrm{e}^{-1}\Big\{&4g_{ij}|\Big[\tfrac12(\gamma^c)_{\alpha\beta}\nabla_c\chi^{\beta i} - \tfrac14 B\chi_\alpha{}^i\Big]\chi^{\alpha j} + 2g_{ij}|\Big[-\tfrac12\nabla^c\phi^i\nabla_c\phi^j + 2F^iF^j\Big] \\
&+g_{ij}|\Big[\psi_d{}^\alpha\nabla^d\, f^i\chi_\alpha{}^j + \epsilon^{abc}(\gamma_c)^\alpha_\beta\psi_a{}^\beta\chi^i_\alpha\nabla_b\phi^j + 2(\gamma^a)_{\alpha\beta}\psi^\beta_a F^j\chi^{\alpha i}\Big] \\
&\quad -2\nabla^2 g_{ij}|\chi^{\alpha i}\chi_\alpha{}^j - \Big[2B + \tfrac12\psi_a{}^\alpha\psi_b{}^\beta\epsilon^{abc}(\gamma_c)_{\alpha\beta}\Big]g_{ij}|\chi^{\alpha i}\chi_\alpha{}^j \\
&\quad +\nabla_\beta g_{ij}|\Big[2(\gamma^c)^{\alpha\beta}\nabla_c\phi^i\chi_\alpha{}^j - 4F^i\chi^{\beta j} - \psi_d{}^\alpha(\gamma^d)_\alpha{}^\beta\chi^{\alpha i}\chi_\alpha{}^j\Big]\Big\} \quad , \quad (2.4.15)
\end{aligned}
$$

$$
\begin{aligned}
\int d^3x\,\mathrm{e}^{-1}Q_{IJ}\Gamma^I_\beta W^{J\beta} &= \int d^3x\,\mathrm{e}^{-1}(\mathcal{D}^2 Q_{IJ}\Gamma^I_\beta W^{J\beta})| \\
= \int d^3x\,\mathrm{e}^{-1}\Big\{&\tfrac23 Q_{IJ}|\lambda^I_\beta\lambda^{\beta J} - \nabla^\alpha Q_{IJ}|(\gamma^c)_{\alpha\beta}A^I_c\lambda^{\beta J} - \tfrac12 Q_{IJ}|A^{aI}\psi_a{}^\alpha\lambda^I_\alpha \\
&-\tfrac12 Q_{IJ}|\epsilon^{abc}(\gamma_a)_\alpha{}^\beta A^I_b\psi_c{}^\alpha\lambda^J_\beta - \tfrac32 Q_{IJ}|\epsilon^{abc}A^I_a f^J_{bc}\Big\} \quad , \quad (2.4.16)
\end{aligned}
$$

$$
\begin{aligned}
\int d^3x\,\mathrm{e}^{-1}W(\Phi) &= \int d^3\,\mathrm{e}^{-1}(\mathcal{D}^2 W)| \\
= \int d^3x\,\mathrm{e}^{-1}\Big\{&-2\nabla^2 W| + \psi_d{}^\alpha(\gamma_d)_\alpha{}^\beta\nabla_\beta W| - \Big[2B + \tfrac12\psi_a{}^\alpha\psi_b{}^\beta\epsilon^{abc}(\gamma_c)_{\alpha\beta}\Big]W|\Big\} \quad . \quad (2.4.17)
\end{aligned}
$$

This component action is completely off-shell supersymmetric. We now put it on-shell by integrating out B and F^i. The equation of motion for F^i leads to:

$$
\begin{aligned}
F^i = \tfrac14 g^{ij}|\Big\{&\frac{\delta W}{\delta\Phi^j}\Big| + \frac{\delta g_{kl}}{\delta\Phi^j}\Big|\chi^{\alpha k}\chi_\alpha{}^l + g^{-2}\frac{\delta h_{IJ}}{\delta\Phi^j}\Big|\lambda^{\alpha I}\lambda_\alpha{}^J \\
&+2\nabla_\alpha g_{jl}|\chi^{\alpha l} + \kappa^{-2}\frac{\delta K}{\delta\Phi^j}\Big|B - g_{jl}(\gamma^a)_{\alpha\beta}\psi^\beta_a\chi^{\alpha l}\Big\} \quad , \qquad (2.4.18)
\end{aligned}
$$

and the equation of motion for B yields:

$$
\begin{aligned}
B = \kappa^2 K|^{-1}\Big\{&g^{-2}h_{IJ}|\lambda^{\alpha I}\lambda_\alpha{}^J - 2W| - g_{ij}|\chi^{\alpha i}\chi_\alpha{}^j \\
&-\kappa^{-2}(\nabla_\alpha K|\psi_d{}^\beta(\gamma^d)_\beta{}^\alpha + 2\nabla^2 K|)\Big\} \quad . \qquad (2.4.19)
\end{aligned}
$$

To be completely general we assume that the coupling functions depend on some combination of matter fields, $\mathcal{F}^a$, thus:

$$
\begin{aligned}
\nabla^2 K| \;&=\; \tfrac12\sum_a\sum_b \frac{\delta^2 K}{\delta\mathcal{F}^b\delta\mathcal{F}^a}\Big|\nabla^\alpha\mathcal{F}^b|\nabla_\alpha\mathcal{F}^a| + \sum_a\frac{\delta K}{\delta\mathcal{F}^a}\Big|\nabla^2\mathcal{F}^a| \\
&\equiv\; \tilde\nabla^2 K| + \frac{\delta K}{\delta\Phi^i}\Big|F^i \quad . \qquad (2.4.20)
\end{aligned}
$$

With this definition, we can substitute for F^i in (2.4.19), leading to:

$$
B = \kappa^2 K|^{-1}\Big[1 + \tfrac12\kappa^{-2}K|^{-1}g^{ij}|\frac{\delta K}{\delta\Phi^i}\Big|\frac{\delta K}{\delta\Phi^j}\Big|\Big]^{-1}\Big\{-2W| - \tfrac12\kappa^{-2}g^{ij}|\frac{\delta W}{\delta\Phi^i}\Big|\frac{\delta K}{\delta\Phi^j}\Big|
$$

$$-2\kappa^{-2}\tilde{\nabla}^2 K| + \Big[- g_{kl}| - \tfrac{1}{2}\kappa^{-2} g^{ij} |\frac{\delta K}{\delta \Phi^i}\Big|\frac{\delta g_{kl}}{\delta \Phi^j}\Big|\Big] \chi^{\alpha k}\chi_\alpha{}^l$$

$$-\kappa^2 g^{ij}|\frac{\delta K}{\delta \Phi^i}\Big|\nabla_\alpha g_{jl}|\chi^{\alpha l} + \Big[g^{-2}h_{IJ}| - \tfrac{1}{2}\kappa^{-2}g^{-2}g^{ij}|\frac{\delta K}{\delta\Phi^i}\Big|\frac{\delta h_{IJ}}{\delta \Phi^j}\Big|\Big]\lambda^{\alpha I}\lambda_\alpha{}^J\Big\} \quad . \tag{2.4.21}$$

This equation for the scalar field B is what is required to obtain the on-shell supersymmetry variation of the gravitino. To see this we begin with the off-shell supersymmetry variation of the gravitino:

$$\begin{aligned} \delta_Q \psi_a{}^\beta &= D_a\epsilon^\beta - \epsilon^\alpha(T_{\alpha a}{}^\beta| + T_{\alpha a}{}^b|\psi_b{}^\beta) - \epsilon^\alpha\psi_a{}^\gamma(T_{\alpha\gamma}{}^\beta| + T_{\alpha\gamma}{}^e|\psi_e{}^\beta) \\ &= D_a\epsilon^\beta - \tfrac{1}{2}\epsilon^\alpha(\gamma_a)_\alpha{}^\beta B - \epsilon^\alpha\psi_a{}^\gamma(\gamma^e)_{\alpha\gamma}\psi_e{}^\beta \quad . \end{aligned} \tag{2.4.22}$$

By converting to curved indices and keeping in mind the variation of $\mathrm{e}_m{}^a$:

$$\delta_Q \mathrm{e}_a{}^m = -[\epsilon^\beta T_{\beta a}{}^d| + \epsilon^\beta\psi_a{}^\gamma T_{\beta\gamma}{}^d|]\mathrm{e}_d{}^m = -\epsilon^\beta\psi_a{}^\gamma(\gamma^d)_{\beta\gamma}\mathrm{e}_d{}^m \quad , \tag{2.4.23}$$

the supersymmetry variation of the gravitino can be put into a more canonical form:

$$\delta_Q\psi_m{}^\beta \quad = \quad D_m\epsilon^\beta - \tfrac{1}{2}\epsilon^\alpha(\gamma_m)_\alpha{}^\beta B \quad . \tag{2.4.24}$$

The other fields have the following supersymmetry transformations:

$$\begin{aligned} \delta_Q \mathrm{e}_m{}^a &= \epsilon^\beta\phi_m^\gamma(\gamma^a)_{\beta\gamma} \quad , \\ \delta_Q B &= \tfrac{1}{4}\epsilon^\alpha(\gamma_a)_{\alpha\gamma}\epsilon^{abc}f_{bc}^\gamma \quad , \\ \delta_Q A_c^I &= -\tfrac{1}{3}\epsilon^\gamma(\gamma_c)_\gamma^\beta\lambda_\beta \quad , \\ \delta_Q\lambda_\alpha^I &= \epsilon^\beta\epsilon^{abc}(\gamma_c)_{\alpha\beta}(\tfrac{3}{4}f_{ab}^I + \tfrac{1}{2}(\gamma_b)_\delta^\gamma\psi_a^\delta\lambda_\gamma^I \quad , \\ \delta_Q\phi^i &= -\epsilon^\alpha\chi_\alpha^i \quad , \\ \delta_Q\chi_\alpha^i &= -\tfrac{1}{2}\epsilon^\beta(\gamma^c)_{\alpha\beta}\nabla_c\phi^i + \epsilon_\alpha F^i \quad , \\ \delta_Q F^i &= -\epsilon^\alpha(\tfrac{1}{2}(\gamma^c)_\alpha^\beta\nabla_c\chi_\beta^i + \tfrac{1}{4}B\chi_\alpha^i) \quad . \end{aligned} \tag{2.4.25}$$

The purely bosonic part of the lagrangian is:

$$\begin{aligned} \mathcal{S}_{Bosonic} \quad = \quad \int d^3x\, \mathrm{e}^{-1}\Big[&- \tfrac{1}{2}\kappa^{-2}B^2 - \tfrac{1}{4}\kappa^{-2}\epsilon^{abc}r_{abc} + \tfrac{9}{2}g^{-2}h_{IJ}|f^{abI}f_{ab}^J - g_{ij}|\nabla^c\phi^i\nabla_c\phi^j \\ &- \tfrac{3}{2}Q_{IJ}|\epsilon^{abc}A_a^I f_{bc}^J - 2\partial_i W|F^i - 2BW| + 4g_{ij}F^iF^j\Big] \quad . \end{aligned} \tag{2.4.26}$$

The equations of motion for B and F^i with $K(\Phi) = 1$ and fermions set to zero are:

$$B = -2\kappa^2 W| \qquad , \qquad F^i = \tfrac{1}{4}g^{ij}\partial_j W| \quad . \tag{2.4.27}$$

Substituting these back into the bosonic Lagrangian we have:

$$\begin{aligned} \mathcal{S}_{Bosonic} = \int d^3x\, \mathrm{e}^{-1}\Big[&- \tfrac{1}{4}\kappa^{-2}\epsilon^{abc}r_{abc} + \tfrac{9}{2}g^{-2}h_{IJ}|f^{abI}f_{ab}^J - g_{ij}|\nabla^c\phi^i\nabla_c\phi^j \\ &-\tfrac{3}{2}Q_{IJ}|\epsilon^{abc}A_a^I f_{bc}^J - (\, \tfrac{1}{4}g^{ij}|\partial_i W|\partial_j W| - 2\kappa^2 W|^2\,)\,\Big] \quad . \end{aligned} \tag{2.4.28}$$

The scalar potential for this theory can be read off from above and is given by

$$V(\phi) = \tfrac{1}{4}g^{ij}\partial_i W|\partial_j W| - 2\kappa^2 W|^2 \quad , \tag{2.4.29}$$

and the on-shell supersymmetry variation of the gravitino takes the form

$$\delta_Q \psi_m{}^{\beta} \;=\; D_m \epsilon^{\beta} \;-\; \kappa^2\, \epsilon^{\alpha} (\gamma_m)_{\alpha}{}^{\beta}\, W| \;\; . \qquad (2.4.30)$$

Thus, we see that the superpotential W determines the scalar potential in the action and appears in the gravitino transformation law. From the form of (2.1.2) and the discussion of the above section, it is clear the issue of an AdS background is described in the usual manner known to superspace practicioners. In the limit $R = \sqrt{\lambda}$, $\Sigma_{\alpha}{}^{b} = 0$, $W_{\alpha}{}^{J} = 0$ and $\widehat{\mathcal{R}}^{ab} = 0$ the commutator algebra in (2.1.2) remains consistent in the form

$$\begin{aligned} [\nabla_{\alpha}\, ,\ \nabla_{\beta}\} &= (\gamma^c)_{\alpha\beta}\, \nabla_c \;-\; \sqrt{\lambda}\, (\gamma^c)_{\alpha\beta}\, \mathcal{M}_c \quad , \\ [\nabla_{\alpha}\, ,\ \nabla_{b}\} &= \tfrac{1}{2} \sqrt{\lambda}\, (\gamma_b)_{\alpha}{}^{\delta}\, \nabla_{\delta} \;\; , \\ [\nabla_{a}\, ,\ \nabla_{b}\} &= -\,\lambda\, \epsilon_{abc}\, \mathcal{M}^c \quad , \end{aligned} \qquad (2.4.31)$$

and clearly the last of these shows that the Riemann curvature tensor is given by $R_{a\,b}{}^{c} = -\,\lambda\, \epsilon_{ab}{}^{c}$. This in turn implies that the curvature scalar $\epsilon^{a\,b\,c}\, R_{a\,b\,c} = -\,6\lambda$, i. e. describes a space of constant negative curvature. Through the equation for B in (2.4.27) we see that

$$\sqrt{\lambda} \;=\; -\,2\,\kappa^2\, < W| > \; . \qquad (2.4.32)$$

Thus, there is a supersymmetry preserving AdS backgound whenever the condition

$$< W| > \;\; < \; 0\, , \qquad (2.4.33)$$

is satisfied. On the otherhand, supersymmetry is broken whenever

$$< W| > \;\; > \; 0\, . \qquad (2.4.34)$$

We will see in the next section that the compactified action is of the form (2.4.28).

3 Compactification of $\mathcal{M}$-Theory on $Spin(7)$ Holonomy Manifolds

In this section, we perform the compactification of the bosonic part of $\mathcal{M}$-theory on a $Spin(7)$ holonomy manifold M_8. Since $Spin(7)$ holonomy manifolds admit only one covariantly constant spinor, we will obtain a theory with $\mathcal{N} = 1$ supersymmetry in three dimensions. We use the following assumptions and conventions. The eight-dimensional manifold M_8 is taken to be compact and smooth[6]. We shall assume that the size of the internal eight-manifold $l_{M_8} = (\mathcal{V}_{M_8})^{1/8}$ is much bigger than the eleven-dimensional Planck length l_{11}. Here $\mathcal{V}_{M_8}$ denotes the volume of the internal manifold.

[6]For an elegant description of such manifolds see [43].

It was shown in [15, 16, 34] that compactifications of $\mathcal{M}$-theory on both conformally Calabi-Yau four-folds and $Spin(7)$ holonomy manifolds should obey the tadpole cancellation condition

$$\frac{1}{4\kappa_{11}^2}\int_{M_8}\hat{F}^{(1)}\wedge\hat{F}^{(1)}+N_2=T_2\,\frac{\chi_8}{24}\,, \tag{3.1}$$

where $\hat{F}^{(1)}$ is the internal part of the background flux, χ_8 is the Euler characteristic of the internal manifold and N_2 represents the number of space-time filling membranes. κ_{11} is the eleven-dimensional gravitational coupling constant, which is related to the membrane tension T_2 by

$$T_2=\left(\frac{2\pi^2}{\kappa_{11}^2}\right)^{1/3}\,. \tag{3.2}$$

Equation (3.1) is important because it restricts the form of the internal manifold as the Euler characteristic is expressed in terms of the internal fluxes. In our computation, we consider the case $N_2=0$.

In the case when the background fluxes are zero, $\hat{F}^{(1)}=0$, the tadpole cancellation condition (3.1) restricts the class of internal manifolds to those which have zero Euler characteristic. In section 3.1, we consider this particular case and we show that no scalar potential for the moduli fields arises under these circumstances. To relax the constraint $\chi_8=0$ we have to consider a non-zero value for the internal background flux $\hat{F}^{(1)}$. Consequently, we will have to use a warped metric ansatz. In section 3.2, we show that the appearance of background fluxes generates a scalar potential for some of the moduli fields appearing in the three-dimensional $\mathcal{N}=1$ low energy effective action. We also check that the form of this scalar potential and the complete action is a particular case of the more general class of models discussed in the previous section.

3.1 Compactification with Zero Background Flux

The effective action for $\mathcal{M}$-theory has the form (see e.g. [20], where all relevant references are provided)

$$S=S_0+S_1+S_2\,. \tag{3.1.1}$$

In this expression S_0 is the bosonic truncation of eleven-dimensional supergravity [44] and S_1 and S_2 represent the leading quantum corrections. The above terms in the action take the following form

$$\begin{aligned}
S_0 &= \frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}}d^{11}x\sqrt{-g}R-\frac{1}{4\kappa_{11}^2}\int_{\mathcal{M}}\left(F\wedge\star F+\frac{1}{3}C\wedge F\wedge F\right)\,,\\
S_1 &= -T_2\int_{\mathcal{M}}C\wedge X_8\,,\\
S_2 &= b_1T_2\int_{\mathcal{M}}d^{11}x\sqrt{-g}\left(J_0-\frac{1}{2}E_8\right)\,,
\end{aligned} \tag{3.1.2}$$

where F is a four-form field strength with potential C and b_1 is a constant

$$b_1 = \frac{1}{(2\pi)^4 3^2 2^{13}} \,. \tag{3.1.3}$$

X_8 is a quartic polynomial of the eleven-dimensional Riemann tensor, whose integral over the internal manifold is related to the Euler characteristic

$$X_8(\mathcal{M}) = \frac{1}{192\,(2\pi)^4}\Big[\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\Big] \,, \qquad \int_{M_8} X_8 \;=\; -\frac{\chi_8}{24} \,. \tag{3.1.4}$$

Furthermore, E_8 and J_0 are quartic polynomials in the eleven-dimensional Riemann tensor, which take the form

$$\begin{aligned} E_8(\mathcal{M}) &= \frac{1}{3!}\,\epsilon^{ABCM_1N_1\ldots M_4N_4}\,\epsilon_{ABCM_1'N_1'\ldots M_4'N_4'}R^{M_1'N_1'}{}_{M_1N_1}\ldots R^{M_4'N_4'}{}_{M_4N_4} \,, & (3.1.5)\\ J_0(\mathcal{M}) &= t^{M_1N_1\ldots M_4N_4}t_{M_1'N_1'\ldots M_4'N_4'}R^{M_1'N_1'}{}_{M_1N_1}\ldots R^{M_4'N_4'}{}_{M_4N_4} + \frac{1}{4}E_8(\mathcal{M}) \,. & (3.1.6)\end{aligned}$$

The tensor t is defined by its contraction with some antisymmetric tensor A by

$$t^{M_1\ldots M_8}A_{M_1M_2}\ldots A_{M_7M_8} = 24\mathrm{tr}A^4 - 6(\mathrm{tr}A^2)^2 \,, \tag{3.1.7}$$

and in general we can define $E_n(M_D)$ for any even n and any D-dimensional manifold M_D $(n \leq D)$

$$\begin{aligned} E_n(M_D) \;=\; & \frac{1}{(D-n)!}\,\epsilon^{M_1\ldots M_{D-n}N_1\ldots N_n}\,\epsilon_{M_1\ldots M_{D-n}N_1'\ldots N_n'} \\ & R^{N_1'N_2'}{}_{N_1N_2}\ldots R^{N_{n-1}'N_n'}{}_{N_{n-1}N_n} \,. \end{aligned} \tag{3.1.8}$$

Our goal is to compactify the action (3.1.1) on a compact $Spin(7)$ holonomy manifold. In order to achieve this we make an ansatz for the eleven-dimensional metric $g^{(11)}_{MN}(x,y)$, which respects the maximal symmetry of the external space (described by the metric $g^{(3)}_{\mu\nu}(x)$, which is not necessarily Minkowski)

$$g^{(11)}_{MN}(x,y) = \begin{pmatrix} g^{(3)}_{\mu\nu}(x) & 0 \\ 0 & g^{(8)}_{mn}(x,y) \end{pmatrix} \,. \tag{3.1.9}$$

Here x represents the external coordinates labelled by $\mu = 0,1,2$, while y represents the internal coordinates labelled by $m = 3,\ldots,10$, and M, N run over the complete eleven-dimensional coordinates. In addition, $g_{mn}(x,y)$ depends on a set of parameters which characterize the possible deformations of the internal metric. These parameters, called moduli, appear as massless scalar fields in the three-dimensional effective action. In other words, an arbitrary vacuum state is characterized by the vacuum expectation values of these moduli fields. In the compactification process we choose an arbitrary vacuum state or equivalently an arbitrary point in moduli space and consider infinitesimal displacements around this point. Consequently, the metric will be

$$g_{mn}(x,y) = \hat{g}_{mn}(y) + \delta g_{mn}(x,y) \,, \tag{3.1.10}$$

where $\hat{g}_{mn}$ is the background metric and δg_{mn} is its deformation. The deformations of the metric are expanded in terms of the zero modes of the Lichnerowicz operator. Furthermore, it was shown in [45], that for a $Spin(7)$ holonomy manifold, the zero modes of the Lichnerowicz operator e_A are in one to one correspondence with the anti-self-dual harmonic four-forms ξ_A of the internal manifold

$$e_{A\,mn}(y) = \frac{1}{6}\,\xi_{A\,mabc}(y)\,\Omega_n{}^{abc}(y)\,, \tag{3.1.11}$$

$$\xi_{A\,abcd}(y) = -\,e_{A\,[a}{}^{m}(y)\Omega_{bcd]m}(y)\,, \tag{3.1.12}$$

where $A = 1, \dots b_4^-$ and Ω is the Cayley calibration of the internal manifold, which in our convention is self-dual. The tensor e^I_{mn} is symmetric and traceless (see [45]). b_4^- is the Betti number that counts the number of anti-self-dual harmonic four-forms of the internal space.

Besides the zero modes of the Lichnerowicz operator there is an additional volume-changing modulus, which corresponds to an overall rescaling of the background metric. So the metric deformations take the following form

$$\delta g_{mn}(x,y) = \phi(x)\,\hat{g}_{mn}(y) + \sum_{A=1}^{b_4^-} \phi^A(x)\,e_{A\,mn}(y)\,, \tag{3.1.13}$$

where ϕ is the radial modulus fluctuation and ϕ^A are the scalar field fluctuations that characterize the deformations of the metric along the directions e_A. Therefore the internal metric has the following expression

$$g_{mn}(x,y) = \hat{g}_{mn}(y) + \phi(x)\,\hat{g}_{mn}(y) + \sum_{A=1}^{b_4^-} \phi^A(x)\,e_{A\,mn}(y)\,. \tag{3.1.14}$$

The three-form potential and the corresponding field strength have fluctuations around their backgrounds $\hat{C}(y)$ and $\hat{F}(y)$ respectively, which in this section are considered to be zero. The fluctuations of the three-form potential are decomposed in terms of the zero modes of the Laplace operator. Taking into account that for $Spin(7)$ holonomy manifolds there are no harmonic one-forms (see (D.2)) the decomposition of the three-form potential has two pieces

$$\delta C(x,y) = \delta C^{(1)}(x,y) + \delta C^{(2)}(x,y) = \sum_{I=1}^{b_2} A^I(x) \wedge \omega_I(y) + \sum_{J=1}^{b_3} \rho^J(x)\,\zeta_J(y)\,, \tag{3.1.15}$$

where ω_I are harmonic two-forms and ζ_J are harmonic three-forms. The set of b_2 vector fields $A^I(x)$ and the set of b_3 scalar fields $\rho^J(x)$ are infinitesimal quantities that characterize the fluctuation of the three-form potential around its background value. The fluctuations of the field strength F are then

$$\delta F(x,y) = \delta F^{(1)}(x,y) + \delta F^{(2)}(x,y) = \sum_{I=1}^{b_2} dA^I(x) \wedge \omega_I(y) + \sum_{J=1}^{b_3} d\rho^J(x) \wedge \zeta_J(y)\,. \tag{3.1.16}$$

Substituting (3.1.14), (3.1.15) and (3.1.16) into S and considering the lowest order contribution in moduli fields we obtain

$$\begin{aligned} S_{3D} &= \frac{1}{2\kappa_3^2}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\Big\{R^{(3)} - 18(\partial_\alpha\phi)(\partial^\alpha\phi) - \sum_{A,B=1}^{b_4^-}\mathcal{L}_{AB}(\partial_\alpha\phi^A)(\partial^\alpha\phi^B) \\ &\quad - \sum_{I,J=1}^{b_3}\mathcal{M}_{IJ}(\partial_\alpha\rho^I)(\partial^\alpha\rho^J) - \sum_{I,J=1}^{b_2}\mathcal{K}_{IJ}\; f^I_{\alpha\beta}f^{J\alpha\beta}\Big\} \;+\; \ldots\,, \end{aligned} \tag{3.1.17}$$

where the ellipses denote higher order terms in moduli fluctuations. κ_3 is the three-dimensional gravitational coupling constant

$$\kappa_3^2 = \mathcal{V}_{M_8}^{-1}\,\kappa_{11}^2\,, \tag{3.1.18}$$

and $\mathcal{V}_{M_8}$ is the volume of the internal manifold

$$\mathcal{V}_{M_8} = \int_{M_8} d^8y\,\sqrt{\hat{g}^{(8)}}\,. \tag{3.1.19}$$

The details of the dimensional reduction of the Einstein-Hilbert term can be found in appendix E.1. The other quantities appearing in (3.1.17) are the field strength f^I of the b_2 $U(1)$ gauge fields A^I

$$f^I_{\alpha\beta} = \partial_{[\alpha}A^I_{\beta]} = \frac{1}{2}(\partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha)\,, \tag{3.1.20}$$

and the metric coefficients for the kinetic terms

$$\mathcal{L}_{AB} = \frac{1}{4\mathcal{V}_{M_8}}\int_{M_8} d^8y\,\sqrt{\hat{g}^{(8)}}\,e_{A\,am}\,e_{B\,bn}\,\hat{g}^{ab}\,\hat{g}^{mn}\,, \tag{3.1.21}$$

$$\mathcal{K}_{IJ} = \frac{3}{2\mathcal{V}_{M_8}}\int_{M_8}\omega_I\wedge\star\,\omega_J\,, \tag{3.1.22}$$

$$\mathcal{M}_{IJ} = \frac{2}{\mathcal{V}_{M_8}}\int_{M_8}\zeta_I\wedge\star\,\zeta_J\,. \tag{3.1.23}$$

With the help of (3.1.11) and (3.1.12) we can rewrite (3.1.21) as follows

$$\mathcal{L}_{AB} = \frac{1}{\mathcal{V}_{M_8}}\int\xi_A\wedge\star\,\xi_B\,. \tag{3.1.24}$$

Note that the Hodge $\star$ operator used in the previous relations is defined with respect to the background metric. As we can see in the zero flux case, the action contains only the gravitational part plus kinetic terms of the massless moduli fields and no scalar potential.

3.2 Compactification with Non-Zero Background Flux

Our goal in this section is to compute the form of the three-dimensional effective action in the presence of non-vanishing fluxes. We begin by decomposing the metric and flux fields and then work out the compactification. We make the following maximally symmetric

ansatz for the metric

$$\widetilde{g}^{(11)}_{MN}(x,y) = \begin{pmatrix} e^{2\Delta(y)/3}\, g^{(3)}_{\mu\nu}(x) & 0 \\ 0 & e^{-\Delta(y)/3}\, g^{(8)}_{mn}(x,y) \end{pmatrix}\,, \tag{3.2.1}$$

where $\Delta(y)$ is the scalar warp factor, $g^{(3)}_{\mu\nu}(x)$ is the metric for the external space and $g^{(8)}_{mn}(x,y)$ has $Spin(7)$ holonomy. Maximal symmetry of the external space restricts the form of the background flux to

$$\begin{aligned} \hat{F}(y) &= \hat{F}^{(1)}(y) + \hat{F}^{(2)}(y)\,, \\ \hat{F}^{(1)}(y) &= \frac{1}{4!}\, F_{mnpq}(y)\, dy^m \wedge dy^n \wedge dy^p \wedge dy^q\,, \\ \hat{F}^{(2)}(y) &= \frac{1}{3!}\, \epsilon_{\alpha\beta\gamma}\partial_m f(y)\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dy^m\,, \end{aligned} \tag{3.2.2}$$

therefore, C has the following background

$$\begin{aligned} \hat{C}(y) &= \hat{C}^{(1)}(y) + \hat{C}^{(2)}(y)\,, \\ \hat{C}^{(1)}(y) &= \frac{1}{3!}\, C_{mnp}(y)\, dy^m \wedge dy^n \wedge dy^p\,, \\ \hat{C}^{(2)}(y) &= -\frac{1}{3!}\, \epsilon_{\alpha\beta\gamma} f(y)\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma\,. \end{aligned} \tag{3.2.3}$$

In addition, $f(y)$ is related to the warp factor $\Delta(y)$ by [34]

$$f(y) = e^{\Delta(y)}\,. \tag{3.2.4}$$

The 5-brane Bianchi identity derived in [34] implies that the warp factor is a small quantity, $\Delta \sim \mathcal{O}(l_{11}^6/l_8^6)$. Further, the tadpole anomaly equation (3.1) implies that the internal flux is also small, $\hat{F}^{(1)} \sim \mathcal{O}(l_{11}^3/l_8^4)$. Consequently, we will consider only the leading order contribution.

Next we start the compactification of the eleven-dimensional action. The Einstein-Hilbert term becomes

$$\begin{aligned} \frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}} d^{11}x\,\sqrt{-\widetilde{g}^{(11)}}\,\widetilde{R}^{(11)} &= \frac{1}{2\kappa_3^2}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{R^{(3)} - 18(\partial_\alpha\phi)\,(\partial^\alpha\phi) \\ &\quad - \sum_{I,J=1}^{b_4^-} \mathcal{L}_{IJ}\,(\partial_\alpha\phi^I)\,(\partial^\alpha\phi^J)\Big\} \;+\; \ldots\,, \end{aligned} \tag{3.2.5}$$

where $\mathcal{L}_{IJ}$ was defined in section 3.1. The details of the dimensional reduction can be found in appendix E.2. The quartic polynomial J_0, (3.1.6), which appears in the definition (3.1.2) of S_2, is the sum of an internal and external polynomial. Further, it can be written in terms of the Weyl tensor [46, 47]. Since the Weyl tensor vanishes in three dimensions we are left with only the internal polynomial $J_0(M_8)$. We shall restrict to compactifications on manifolds with $J_0(M_8) = 0$. It is rather possible that this holds for a generic $Spin(7)$

holonomy manifold, however this needs to be evaluated in more detail [48]. The final piece of S_2 involves the quartic polynomial E_8. For a product space $M_3 \times M_8$ we have [20]

$$E_8(M_3 \times M_8) = -E_8(M_8) + 4\, E_2(M_3)\, E_6(M_8)\,, \tag{3.2.6}$$

where $E_2(M_3) = -2R^{(3)}$. Therefore

$$b_1 T_2 \int_{\mathcal{M}} d^{11}x \sqrt{-g} \left(J_0 - \frac{1}{2} E_8 \right) = \int_{M_3} d^3x \sqrt{-g^{(3)}}\, T_2\, \frac{\chi_8}{24} \;+\; \ldots\,, \tag{3.2.7}$$

where we have used the fact that the Euler characteristic of the internal manifold is

$$\chi_8 = 12 b_1 \int_{M_8} d^8y \sqrt{\hat{g}^{(8)}}\, E_8(M_8)\,, \tag{3.2.8}$$

and we have neglected the subleading contribution from the second term of (3.2.6). This concludes the analysis of the terms in S that only depend on the metric.

The remaining terms in S consist of the kinetic term for C, the Chern-Simons term, and the tadpole anomaly term S_1. The expressions (3.1.16) and (3.2.2) of the field strength F imply that

$$\begin{aligned} \int_{\mathcal{M}} F \wedge \star F \;\; &= \;\; \int_{\mathcal{M}} \hat{F}^{(1)} \wedge \star \hat{F}^{(1)} + \int_{\mathcal{M}} \hat{F}^{(2)} \wedge \star \hat{F}^{(2)} \\ &+ \;\; \int_{\mathcal{M}} \delta F^{(1)} \wedge \star \delta F^{(1)} + \int_{\mathcal{M}} \delta F^{(2)} \wedge \star \delta F^{(2)}\,, \end{aligned} \tag{3.2.9}$$

where the second term is subleading and will be neglected. To leading order, the last two terms in the above sum can be expressed as

$$\begin{aligned} &\frac{1}{4\kappa_{11}^2} \int_{\mathcal{M}} [\delta F^{(1)} \wedge \star \delta F^{(1)} + \delta F^{(2)} \wedge \star \delta F^{(2)}] \\ &= \frac{1}{2\kappa_3^2} \int_{M_3} d^3x \sqrt{-g^{(3)}} \Big\{ \sum_{I,J=1}^{b_3} \mathcal{M}_{IJ}\, (\partial_\alpha \rho^I)(\partial^\alpha \rho^J) \\ &+ \sum_{I,J=1}^{b_2} \mathcal{K}_{IJ}\, f^I_{\alpha\beta}\, f^{J\alpha\beta} \Big\}, \end{aligned} \tag{3.2.10}$$

where f^I, $\mathcal{K}_{IJ}$ and $\mathcal{M}_{IJ}$ were defined in section 3.1. Due to specific structure of $C(x, y)$ and $F(x, y)$ given in equations (3.1.15), (3.1.16), (3.2.2) and (3.2.3) the Chern-Simons term will have the following form to leading order in moduli field fluctuations

$$\int_{\mathcal{M}} C \wedge F \wedge F = 3 \int_{\mathcal{M}} \hat{C}^{(2)} \wedge \hat{F}^{(1)} \wedge \hat{F}^{(1)} + 2 \int_{\mathcal{M}} \delta C^{(1)} \wedge \delta F^{(1)} \wedge \hat{F}^{(1)} \;+\; \ldots\,. \tag{3.2.11}$$

Since the first term in (3.2.11) cancels the tadpole anomaly term, S_1, the sum of the Chern-Simons term and S_1 is

$$\frac{1}{12\kappa_{11}^2} \int_{\mathcal{M}} C \wedge F \wedge F + T_2 \int_{\mathcal{M}} C \wedge X_8 = \frac{1}{6\kappa_{11}^2} \int_{\mathcal{M}} \delta C^{(1)} \wedge \delta F^{(1)} \wedge \hat{F}^{(1)} \;+\; \ldots\,. \tag{3.2.12}$$

Using (3.1.15) and (3.1.16) we obtain

$$\frac{1}{6\kappa_{11}^2}\int_{\mathcal{M}}\delta C^{(1)}\wedge\delta F^{(1)}\wedge\hat{F}^{(1)}=\frac{1}{2\kappa_3^2}\sum_{I,J=1}^{b_2}\mathcal{E}_{IJ}\int_{M_3}A^I\wedge dA^J\,, \tag{3.2.13}$$

where we have defined

$$\mathcal{E}_{IJ}=\frac{1}{3\mathcal{V}_{M_8}}\int_{M_8}\omega^I\wedge\omega^J\wedge\hat{F}^{(1)}\,. \tag{3.2.14}$$

The coefficient (3.2.14) is proportional to the internal flux and this is the reason why we did not obtain such a term in section 3.1. We have completed the compactification of S. Using the above formulas we obtain to leading order in moduli fields the following expression for the low energy effective action

$$\begin{aligned} S_{3D} \;=\; & \frac{1}{2\kappa_3^2}\int_{M_3}d^3x\,\sqrt{-g^{(3)}}\,\Big\{R^{(3)}-18(\partial_\alpha\phi)(\partial^\alpha\phi)-\sum_{I,J=1}^{b_4^-}\mathcal{L}_{IJ}(\partial_\alpha\phi^I)(\partial^\alpha\phi^J)\\ & -\sum_{I,J=1}^{b_3}\mathcal{M}_{IJ}\,(\partial_\alpha\rho^I)(\partial^\alpha\rho^J)-\sum_{I,J=1}^{b_2}\mathcal{K}_{IJ}\,f^I_{\alpha\beta}\,f^{J\alpha\beta}\\ & -\sum_{I,J=1}^{b_2}\mathcal{E}_{IJ}\,\epsilon^{\mu\nu\sigma}A^I_\mu\,f^J_{\nu\sigma}-V\Big\}+\dots\,, \end{aligned} \tag{3.2.15}$$

where the scalar potential V is

$$V=\frac{1}{2\mathcal{V}_{M_8}}\int_{M_8}\hat{F}^{(1)}\wedge\star\hat{F}^{(1)}-2\kappa_3^2\,T_2\,\frac{\chi_8}{24}\,. \tag{3.2.16}$$

This potential is a particular case of the more general construction presented in the previous section (2.4.29). Let us elaborate this in detail.

First, we will show that the scalar potential (3.2.16) can be written in terms of the superpotential

$$W=\int_{M_8}\hat{F}^{(1)}\wedge\Omega\,. \tag{3.2.17}$$

The form of the superpotential W was conjectured in [37]. This conjecture has been checked recently in [35, 36]. More explicitly in [36] the supersymmetry transformation for the gravitino (2.4.30) was used to obtain the form of the superpotential. Using the anomaly cancellation condition (3.1), the scalar potential becomes

$$V=\frac{1}{\mathcal{V}_{M_8}}\int_{M_8}\hat{F}^{(1)}_-\wedge\star\hat{F}^{(1)}_-\;\;, \tag{3.2.18}$$

where

$$\hat{F}^{(1)}_-=\frac{1}{2}\left[\hat{F}^{(1)}-\star\hat{F}^{(1)}\right]\,, \tag{3.2.19}$$

is the anti-self-dual part of the internal flux $\hat{F}^{(1)}$. Using the definition (3.1.24) for $\mathcal{L}_{AB}$ we can obtain the functional dependence of the scalar potential V in terms of the superpotential (3.2.17)

$$V[W]=\sum_{A,B=1}^{b_4^-}\mathcal{L}^{AB}\,D_AW\,D_BW\,, \tag{3.2.20}$$

where $\mathcal{L}^{AB}$ is the inverse matrix of $\mathcal{L}_{AB}$ and we have introduced the operator

$$D_A\Omega = \partial_A\Omega + K_A\Omega \,. \tag{3.2.21}$$

As shown in the appendix D (see D.6) if the D_A operator acts on the Cayley calibration the result is an anti-self-dual harmonic four-form.

What we observe from (3.2.20) is that the external space is restricted to three-dimensional Minkowski because the scalar potential is a perfect square, in agreement with [35]. Furthermore, when $D_A W = 0$ the scalar potential vanishes. This gives us a set of b_4^- equations for $b_4^- + 1$ fields, so that the radial modulus is not fixed at this level. Its rather possible that non-perturbative effects will lead to a stabilization of this field, as in [49].

A few remarks are in order before we can compare the compactified action to the supergravity action. For a consistent analysis, we must take into account all of the kinetic terms for the metric moduli. Furthermore, the scalar potential (3.2.20) does not seem to be a special case of (2.4.29). The discrepancy arises for two reasons. First, in the general case the superpotential may depend on all of the scalar fields existing in the theory and the summation in (2.4.29) is taking into account all of these scalars, whereas in the compactified version the superpotential depends only on the metric moduli

$$\partial_i W \neq 0 \qquad i = 0,\ 1, \ldots b_4^- \,, \tag{3.2.22}$$

where "0" labels the radial modulus. Second, as described in (D.4) the superpotential has a very special radial modulus dependence in the sense that

$$\partial_0 W = 2W \,, \tag{3.2.23}$$

and this is the reason why the summation in (3.2.20) does not include the radial modulus. Keeping these remarks in mind we proceed to show that the result coming from compactification is a particular case of the general supergravity analysis.

We begin by rescaling some of the fields in the supergravity action

$$\begin{aligned} 2\kappa_3^2 \; g_{ij}| &= L_{ij} \,, \\ \frac{1}{2\kappa_3^2} \; g^{ij}| &= L^{ij} \,, \\ \kappa_3^2 \; W| &= \widetilde{W} \,. \end{aligned} \tag{3.2.24}$$

The relevant terms in the supergravity action (2.4.28) can be written as

$$\begin{aligned} &\int_{M_3} d^3x \, \sqrt{-g^{(3)}} \, \Big\{ - g_{ij} \; \partial_\alpha \bar{\phi}^i \; \partial^\alpha \bar{\phi}^j - [\, \tfrac{1}{4} g^{ij} \; \bar{\partial}_i W \; \bar{\partial}_j W - 2\kappa_3^2 \; W^2 \,] \Big\} \\ &= \frac{1}{2\kappa_3^2} \int_{M_3} d^3x \, \sqrt{-g^{(3)}} \, \Big\{ - L_{ij} \; \partial_\alpha \bar{\phi}^i \; \partial^\alpha \bar{\phi}^j - [\, L^{ij} \; \bar{\partial}_i \widetilde{W} \; \bar{\partial}_j \widetilde{W} - 4 \; \widetilde{W}^2 \,] \Big\} . \end{aligned} \tag{3.2.25}$$

In the above equation the indices $i,\ j = 0,\ 1, \ldots b_4^-$. In what follows we will drop the label "0" from the radion $\phi_0 = \phi$ and the derivative with respect to it $\partial_0 = \partial$ and we will denote

with $A,\ B\ldots$ the remaining set of indices, i.e. $A,\ B = 1,\ldots,b_4^-$. We have placed bars on the scalar fields and their derivatives in (3.2.25) to avoid confusion since we require one more field redefinition.

The relevant terms from the compactified action have the following form

$$\begin{aligned}
&\frac{1}{2\kappa_3^2}\int_{M_3} d^3x\sqrt{-g^{(3)}}\Big\{-18(\partial\phi)^2 - \mathcal{L}_{AB}(\partial_\alpha\phi^A)(\partial^\alpha\phi^B) - \mathcal{L}^{AB}D_AWD_BW\ \Big\}\\
&\qquad = \int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{-18(\partial_\alpha\phi)(\partial^\alpha\phi) - \mathcal{L}_{AB}(\partial_\alpha\phi^A)(\partial^\alpha\phi^B)\\
&\qquad\qquad -[\mathcal{L}^{AB}(\partial_AW)(\partial_BW) + 4\mathcal{L}^A(\partial_AW)W + \mathcal{L}W^2]\ \Big\}.
\end{aligned}\tag{3.2.26}$$

In the above equation we have used the expression (3.2.21) for D_A and we have introduced $\mathcal{L}^A = \mathcal{L}^{AB}K_B$ and $\mathcal{L} = \mathcal{L}^{AB}K_AK_B$. In order to make the comparison between (3.2.25) and (3.2.26), we have to redefine the fields in (3.2.25) in the following manner

$$\begin{aligned}
\phi &= L_{00}\bar{\phi} + L_{0A}\bar{\phi}^A\,,\\
\phi^A &= \bar{\phi}^A\,.
\end{aligned}\tag{3.2.27}$$

Keeping track that ϕ is the radial modulus, we obtain the following form for the relevant terms of the supergravity action

$$\begin{aligned}
&\frac{1}{2\kappa_3^2}\int_{M_3} d^3x\sqrt{-g^{(3)}}\Big\{-L_{ij}(\partial_\alpha\bar{\phi}^i)(\partial^\alpha\bar{\phi}^j) - [L^{ij}(\bar{\partial}_i\widetilde{W})(\bar{\partial}_j\widetilde{W}) - 4\widetilde{W}^2]\ \Big\}\\
&= \frac{1}{2\kappa_3^2}\int_{M_3} d^3x\sqrt{-g^{(3)}}\Big\{-\frac{1}{L_{00}}(\partial_\alpha\phi)(\partial^\alpha\phi) - (L_{AB} - \frac{L_{0A}L_{0B}}{L_{00}})(\partial_\alpha\phi^A)(\partial^\alpha\phi^B)\\
&\quad -[L^{AB}(\partial_A\widetilde{W})(\partial_B\widetilde{W}) + 4L^{0A}L_{00}\widetilde{W}(\partial_A\widetilde{W}) + 4(L^{00}L_{00}^2 + 3L_{00} + 1)\widetilde{W}^2]\Big\}.
\end{aligned}\tag{3.2.28}$$

Surprisingly, we have that

$$(L_{AB} - \frac{L_{0A}L_{0B}}{L_{00}})L^{BC} = \delta_A^C,\tag{3.2.29}$$

and as a consequence we can perform the following identifications

$$\begin{aligned}
L_{00} &= \frac{1}{18}\,,\\
L_{AB} - \frac{L_{0A}L_{0B}}{L_{00}} &= \mathcal{L}_{AB}\,,\\
L^{AB} &= \mathcal{L}^{AB}\,,\\
L^{0A}L_{00} &= \mathcal{L}^{AB}K_B\,,\\
4(L^{00}L_{00}^2 + 3L_{00} + 1) &= \mathcal{L}^{AB}K_AK_B\,.
\end{aligned}\tag{3.2.30}$$

With these identifications, both actions are seen to coincide. The remaining kinetic terms and the Chern-Simons terms that were left in the actions (2.4.28) and (3.2.15) can be easily identified and we conclude that the compactified action is in perfect agreement with the general supergravity action. $\mathcal{M}$-theory compactified on manifolds with $Spin(7)$ holonomy produces a low energy effective action that corresponds to a particular case of the minimal three dimensional supergravity coupled with matter.

4 Summary and Open Questions

In this paper we have derived the general form of 3D, $\mathcal{N} = 1$ supergravity coupled to matter. The off-shell component action is the sum of (2.4.13-2.4.17), the on-shell bosonic action is given in (2.4.28), and the supersymmetry variation of the gravitino, (2.4.30), was shown to be proportional to the superpotential. The latter statement was an important ingredient in order to check [36] the form of the superpotential for compactifications of $\mathcal{M}$-theory on $Spin(7)$ holonomy manifolds conjectured in [37]. We have also performed the Kaluza-Klein compactification of $\mathcal{M}$-theory on a $Spin(7)$ holonomy manifold with and without fluxes. When fluxes are included, we generate a scalar potential for moduli fields. This scalar potential can be expressed in terms of the superpotential, (3.2.17). Interestingly, the potential (3.2.20) is a perfect square, so that only compactifications to three-dimensional Minkowski space can be obtained in agreement with [35]. It is plausible that non-perturbative effects will modify this result to three-dimensional de-Sitter space along the lines of [49]. This will be an interesting question for the future.

Another interesting issue we addressed is the duality between a strongly coupled gauge theory and a weakly coupled supergravity theory. Recall that the supergravity dual to the 4D confining gauge theory given by Polchinski and Strassler [31] has yet to be found to all orders in perturbation theory. This verification could be obtained by considering a generalization of the compactifications of $\mathcal{M}$-theory on eight manifolds of [15]. Work in this direction has been done recently in [50, 51, 52], though the complete supergravity dual to all orders is still lacking. In a similar vein, it would be interesting to find gauge theories which are dual to compactifications of $\mathcal{M}$-theory on $Spin(7)$ holonomy manifolds. These theories would be 3D conformal gauge theories with $\mathcal{N} = 1$ supersymmetry. It may then be possible to deform the $Spin(7)$-manifold to obtain a confining gauge theory.

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A Notations and Conventions

A.1 3D Supergravity

We use lower case Latin letters for three-vector indices and Greek letters for spinor indices. Supervector indices are denoted by capital Latin letters A, M. We further employ the early late convention: letters at the beginning of the alphabet denote tangent space indices while letters from the middle of the alphabet denote coordinate indices. The spinor metric is defined through:

$$C_{\mu\nu}C^{\sigma\tau} \;=\; \delta_\mu{}^\sigma\,\delta_\nu{}^\tau \;-\; \delta_\mu{}^\tau\,\delta_\nu{}^\sigma \;\equiv\; \delta_\mu{}^{[\sigma}\,\delta_\nu{}^{\tau]} \quad , \tag{A.1.1}$$

and is used to raise and lower spinor indices via:

$$\theta_\nu \;=\; \theta^\mu\, C_{\mu\nu} \quad , \quad \theta^\mu \;=\; C^{\mu\nu}\,\theta_\nu \quad . \tag{A.1.2}$$

Some other conventions:

$$diag(\,\eta_{ab}\,) \;=\; (-1,1,1) \quad , \quad \epsilon_{abc}\,\epsilon^{def} \;=\; -\,\delta_{[a}{}^d\delta_b{}^e\delta_{c]}{}^f, \quad \epsilon^{012} = +1 \quad . \tag{A.1.3}$$

The γ-matrices are defined through:

$$(\gamma^a)_\alpha{}^\gamma(\gamma^b)_\gamma{}^\beta = \eta^{ab}\delta_\alpha{}^\beta + \epsilon^{abc}(\gamma_c)_\alpha{}^\beta \quad , \tag{A.1.4}$$

and satisfy the Fierz identities:

$$(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} \;=\; -\,\delta_{(\alpha}{}^\gamma\delta_{\beta)}{}^\delta \;=\; -\,(\gamma^a)_{(\alpha}{}^\gamma(\gamma_a)_{\beta)}{}^\delta \quad , \tag{A.1.5}$$

$$\epsilon^{abc}\,(\gamma_b)_{\alpha\beta}(\gamma_c)_\gamma{}^\delta \;=\; C_{\alpha\gamma}(\gamma^a)_\beta{}^\delta + (\gamma^a)_{\alpha\gamma}\delta_\beta{}^\delta \quad . \tag{A.1.6}$$

For the Levi-Cevita symbol, we have the contractions:

$$\begin{aligned}
\epsilon^{abc}\epsilon_{def} &= -\delta_{[d}{}^a\delta_e{}^b\delta_{f]}^{\,c} \quad , \\
\epsilon^{abc}\epsilon_{dec} &= -\delta_{[d}{}^a\delta_{e]}^{\,b} \quad , \\
\epsilon^{abc}\epsilon_{dbc} &= -2\delta_d{}^a \quad .
\end{aligned} \tag{A.1.7}$$

The Lorentz rotation generator is realized in the following manner:

$$e^{-\frac{1}{2}\lambda_{ab}\mathcal{M}^{ab}} = e^{-\frac{1}{2}\epsilon_{abc}\lambda^c\frac{1}{4}\gamma^{[a}\gamma^{b]}} = e^{+\frac{1}{2}\lambda^c\gamma_c} \quad . \tag{A.1.8}$$

Infinitesimally, the action of the Lorentz generator is:

$$\begin{aligned}
[\;\mathcal{M}_a\,,\,\varphi(x)\;] &= 0 \quad , \\
[\;\mathcal{M}_a\,,\,\rho_\alpha(x)\,] &= \tfrac{1}{2}\,(\gamma_a)_\alpha{}^\beta\,\rho_\beta(x) \quad , \\
[\;\mathcal{M}_a\,,\,A_b(x)\,] &= \epsilon_{bac}\,A^c(x) \quad .
\end{aligned} \tag{A.1.9}$$

Some useful identities:

$$X_{[\alpha\beta]} = -C_{\alpha\beta}X^\gamma{}_\gamma \quad , \tag{A.1.10}$$

$$T_\gamma C_{\beta\delta} + T_\beta C_{\delta\gamma} + T_\delta C_{\gamma\beta} = 0 \quad . \tag{A.1.11}$$

A.2 Differential Forms

If α_p is a p-differential form then its expansion in components is

$$\alpha_p = \frac{1}{p!}\,\alpha_{m_1,\dots,m_p}\,dx^{m_1}\wedge\ldots\wedge dx^{m_p}\,. \tag{A.2.1}$$

Let us consider the wedge product between a p-differential form α_p and a q-differential form β_q. $\alpha_p\wedge\beta_q$ is a $(p+q)$-differential form, so

$$\alpha_p\wedge\beta_q = \frac{1}{(p+q)!}\,(\alpha_p\wedge\beta_q)_{m_1,\dots,m_{p+q}}\,dx^{m_1}\wedge\ldots\wedge dx^{m_{p+q}}\,. \tag{A.2.2}$$

On the other hand, by definition

$$\alpha_p\wedge\beta_q = \frac{1}{p!\,q!}\,\alpha_{[m_1,\dots,m_p}\,\beta_{m_{p+1},\dots,m_{p+q}]}\,dx^{m_1}\wedge\ldots\wedge dx^{m_{p+q}}\,, \tag{A.2.3}$$

therefore

$$(\alpha_p\wedge\beta_q)_{m_1,\dots,m_{p+q}} = \frac{(p+q)!}{p!\,q!}\,\alpha_{[m_1,\dots,m_p}\,\beta_{m_{p+1},\dots,m_{p+q}]}\,. \tag{A.2.4}$$

The definition for the exterior derivation is

$$d\alpha_p = \frac{1}{p!}\,\partial_{[m_1}\,\alpha_{m_2,\dots,m_{p+1}]}\,dx^{m_1}\wedge\ldots\wedge dx^{m_{p+1}}\,. \tag{A.2.5}$$

But $d\alpha_p$ is a $(p+1)$-form

$$d\alpha_p = \frac{1}{(p+1)!}\,(d\alpha_p)_{m_1,\dots,m_{p+1}}\,dx^{m_1}\wedge\ldots\wedge dx^{m_p}\,, \tag{A.2.6}$$

therefore

$$(d\alpha_p)_{m_1,\dots,m_{p+1}} = (p+1)\,\partial_{[m_1}\,\alpha_{m_2,\dots,m_{p+1}]}\,. \tag{A.2.7}$$

The Hodge $\star$ operator of some p-form on a real n-dimensional manifold is defined as

$$\star\ \alpha_p = \frac{\sqrt{g}}{p!(n-p)!}\,\alpha_{k_1\dots k_p}\,g^{k_1m_1}\dots g^{k_pm_p}\,\epsilon_{m_1\dots m_pm_{p+1}\dots m_n}\,dx^{m_{p+1}}\wedge\ldots\wedge dx^{m_n}\,, \tag{A.2.8}$$

where

$$\epsilon_{1\dots n} = +1\,. \tag{A.2.9}$$

Regarding the integration of some p-form α_p on a p-cycle $\mathcal{C}_p$ we have that

$$\int_{\mathcal{C}_p}\alpha_p = \frac{1}{p!}\int_{\mathcal{C}_p}\alpha_{m_1\dots m_p}dx^{m_1}\wedge\ldots\wedge dx^{m_p}\,. \tag{A.2.10}$$

We can also introduce an inner product on the space of real p-forms defined on a n-dimensional manifold $\mathcal{M}$

$$\langle\alpha_p,\,\beta_p\rangle = \int_{\mathcal{M}}\alpha_p\wedge\star\beta_p = \frac{1}{p!}\int_{\mathcal{M}}\alpha_{m_1\dots m_p}\,\beta^{m_1\dots m_p}\,\sqrt{g}\,dx^1\wedge\ldots\wedge dx^n\,. \tag{A.2.11}$$

B Derivation of Fierz Identities

Choosing the real basis:

$$\gamma^0 = i\sigma^2 \quad , \quad \gamma^1 = \sigma^1 \quad , \quad \gamma^2 = \sigma^3 \quad . \tag{B.1}$$

We can show by explicit substitution that:

$$(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} = -\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \quad . \tag{B.2}$$

Basis free, we can derive:

$$\begin{aligned}(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} &= (\gamma^a)_{\alpha}{}^{\epsilon}(\gamma_a)_{\eta}{}^{\delta}C_{\epsilon\beta}C^{\gamma\eta} = (\gamma^a)_{\alpha}{}^{\epsilon}(\gamma_a)_{\eta}{}^{\delta}\delta_{[\epsilon}{}^{\gamma}\delta_{\beta]}{}^{\eta} \\ &= (\gamma^a)_{\alpha}{}^{\gamma}(\gamma_a)_{\beta}{}^{\delta} - \delta_{\beta}{}^{\gamma}(\gamma^a\gamma_a)_{\alpha}{}^{\delta} \\ &= \tfrac{1}{2}(\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} - \tfrac{3}{2}\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \quad .\end{aligned} \tag{B.3}$$

Using this result and (B.2) we also have:

$$(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} = -(\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} \quad . \tag{B.4}$$

The second Fierz identity can be derived directly from the defining relation (A.1.4):

$$\Big\{(\gamma^a)_{\gamma}{}^{\sigma}(\gamma^b)_{\sigma}{}^{\delta} = \eta^{ab}\delta_{\gamma}{}^{\delta} + \epsilon^{abc}(\gamma_c)_{\gamma}{}^{\delta}\Big\}(\gamma_b)_{\alpha\beta} \quad . \tag{B.5}$$

Using (B.2) we can simplify this relation:

$$\begin{aligned}\epsilon^{abc}(\gamma_b)_{\alpha\beta}(\gamma_c)_{\gamma}{}^{\delta} &= (\gamma^a)_{\gamma\sigma}\delta_{(\alpha}{}^{\sigma}\delta_{\beta)}{}^{\delta} - (\gamma^a)_{\alpha\beta}\delta_{\gamma}{}^{\gamma} \\ &= (\gamma^a)_{\alpha\gamma}\delta_{\beta}{}^{\delta} + (\gamma^a)_{\beta[\gamma}\delta_{\alpha]}{}^{\delta} \\ &= (\gamma^a)_{\alpha\gamma}\delta_{\beta}{}^{\delta} + C_{\alpha\gamma}(\gamma^a)_{\beta}{}^{\delta} \quad .\end{aligned} \tag{B.6}$$

A consequence of this identity is:

$$(\gamma_{[c})_{(\alpha}{}^{\delta}(\gamma_{d]})_{\beta)}{}^{\sigma} = -2\epsilon_{acd}C^{\delta\sigma}(\gamma^a)_{\alpha\beta} \quad . \tag{B.7}$$

C Verification of Three-Dimensional Supergravity Covariant Derivative Algebra

The algebra of supergravity covariant derivatives given in the literature is not written in our conventions, and does not contain the gauge fields. To get the correct algebra we take the form given in the literature with arbitrary coefficients and add the superfield strengths $\mathcal{F}_{\alpha\underline{b}}$ and $\mathcal{F}^{\underline{c}I}$ associated with the $U(1)$ gauge theory:

$$\begin{aligned}[\nabla_\alpha \, , \, \nabla_\beta\} &= (\gamma^{\underline{c}})_{\alpha\beta}\,\nabla_{\underline{c}} \; - \; (\gamma^{\underline{c}})_{\alpha\beta}R\,\mathcal{M}_{\underline{c}} \quad , \\ [\nabla_\alpha \, , \, \nabla_{\underline{b}}\} &= \; - a(\gamma_{\underline{b}})_{\alpha}{}^{\delta}R\nabla_{\delta} \; + \; [-2(\gamma_{\underline{b}})_{\alpha}{}^{\delta}\Sigma_{\delta}{}^{\underline{d}} \; + \; b\tfrac{4}{3}(\gamma_{\underline{b}}\gamma^{\underline{d}})_{\alpha}{}^{\epsilon}(\nabla_{\epsilon}R)]\mathcal{M}_{\underline{d}}\end{aligned}$$

$$
\begin{aligned}
& + \; c(\nabla_\alpha R)\mathcal{M}_{\underline{b}} + \mathcal{F}^I_{\alpha \underline{b}} t_I , \\
[\nabla_{\underline{a}} \; , \; \nabla_{\underline{b}} \} \;\; = \;\; & +2\epsilon_{\underline{abc}}[\; d\Sigma^{\alpha \underline{c}} + e\tfrac{2}{3}(\gamma^{\underline{c}})^{\alpha\beta}(\nabla_\beta R) \;]\nabla_\alpha \\
& +\epsilon_{\underline{abc}}[\; \widehat{\mathcal{R}}^{\underline{cd}} \; + \; \tfrac{2}{3}\eta^{\underline{cd}}(f\nabla^2 R \; + \; g\tfrac{3}{2}R^2) \;]\mathcal{M}_{\underline{d}} \\
& +\epsilon_{\underline{abc}}\mathcal{F}^{\underline{c}I} t_I \quad ,
\end{aligned} \tag{C.1}
$$

where $\widehat{\mathcal{R}}^{ab} - \widehat{\mathcal{R}}^{ba} = \eta_{\underline{ab}}\widehat{\mathcal{R}}^{\underline{ab}} = (\gamma_{\underline{d}})^{\alpha\beta}\Sigma_\beta{}^{\underline{d}} = 0$ and

$$
\nabla_\alpha \Sigma_\beta{}^f = -\tfrac{1}{4}(\gamma^e)_{\alpha\beta}\widehat{\mathcal{R}}_e{}^f + \tfrac{1}{6}[C_{\alpha\beta}\eta^{fd} + \tfrac{1}{2}\epsilon^{fde}(\gamma_e)_{\alpha\beta}]\nabla_d R \quad . \tag{C.2}
$$

By checking the Bianchi identities,we will set the coefficients and derive constraints on the new superfield strengths as in (C.2). The Bianchi identity $[[\nabla_{(\alpha}, \nabla_\beta\}, \nabla_{\gamma)}\} = 0$ looks like:

$$
\begin{aligned}
[[\nabla_{(\alpha}, \nabla_\beta\}, \nabla_{\gamma)}\} = -(\gamma^c)_{\alpha\beta}\Big\{ & (\tfrac{1}{2} - a)(\gamma_c)_\gamma{}^\delta R\nabla_\delta + \mathcal{F}^I_{\gamma c} t_I + (c-1)(\nabla_\gamma R)\mathcal{M}_c \\
& +(\gamma_c)_\gamma{}^\delta[-2\Sigma^d_\delta + \tfrac{4}{3}b(\gamma^d)_\delta{}^\epsilon(\nabla_\epsilon R)]\mathcal{M}_d \Big\} + [\beta\gamma\alpha] + [\gamma\alpha\beta] \quad .
\end{aligned} \tag{C.3}
$$

This equation is satisfied if $c = 1$ and

$$
(\gamma^c)_{(\alpha\beta}\mathcal{F}^I_{\gamma)c} = 0 \qquad \Rightarrow \qquad \mathcal{F}^I_{\gamma c} = \tfrac{1}{3}(\gamma_c)_\gamma{}^\alpha W^I_\alpha \quad . \tag{C.4}
$$

The identity $[\{\nabla_\alpha, \nabla_\beta\}, \nabla_c\,] + \{[\nabla_c, \nabla_{(\alpha}], \nabla_{\beta)}\} = 0$ is quite complicated, so we restrict our attention to one algebra element at a time. The terms proportional to t_I are:

$$
(\gamma^d)_{\alpha\beta}\epsilon_{dce}\mathcal{F}^{eI} - \tfrac{1}{3}(\gamma_c)_{(\alpha}{}^\delta\nabla_{\beta)}W^I_\delta = 0 \quad . \tag{C.5}
$$

Multiplying by $(\gamma^c)^{\alpha\beta}$ implies $\nabla^\delta W^I_\delta = 0$. Multiplying by $(\gamma_a)^{\alpha\beta}$ and antisymmetrizing over a and c leads to:

$$
\mathcal{F}^{eI} = \tfrac{1}{3}(\gamma^e)_\beta{}^\delta\nabla^\beta W^I_\delta \quad . \tag{C.6}
$$

Terms proportional to ∇_a are:

$$
-(\gamma^d)_{\alpha\beta}\epsilon_{cde}R\nabla^e + a(\gamma_c)_{(\alpha}{}^\delta(\gamma^d)_{\beta)\delta}R\nabla_d = 0 \quad . \tag{C.7}
$$

Which means $a = -\frac{1}{2}$. Continuing to the terms proportional to $\Sigma_{\beta c}\nabla_\alpha$:

$$
2d\epsilon_{dce}(\gamma^d)_{\alpha\beta}\Sigma^{\delta e}\nabla_\delta + (\gamma_c)_{(\alpha}{}^\delta(\gamma_d)_{\beta)}{}^\sigma\Sigma^d_\delta\nabla_\sigma = 0 \quad . \tag{C.8}
$$

Using (B.7) and the fact that $\Sigma_{\alpha c}$ is gamma traceless, we see that $d = -1$. The terms proportional to $R\nabla_\alpha$ are:

$$
\begin{aligned}
\mathcal{J}_{\alpha\beta\gamma}\nabla^\gamma := \Big\{ & \tfrac{4}{3}eC_{\alpha\gamma}(\gamma_c)_b{}^\rho(\nabla_\rho R) + \tfrac{4}{3}e(\gamma_c)_{\gamma\alpha}(\nabla_\beta R + (\gamma_c)_{\gamma(\alpha}(\nabla_{\beta)}R) \\
& +\tfrac{4}{3}b(\gamma_c)_{\alpha\beta}(\nabla_\gamma R) + \tfrac{2}{3}b(\gamma_c)_{\gamma(\alpha}(\nabla_{\beta)}R)\Big\}\; \nabla^\gamma \quad .
\end{aligned} \tag{C.9}
$$

Therefore $\mathcal{J}_{\alpha\beta\gamma} = 0$. $\mathcal{J}_{\alpha\beta\gamma}$ is symmetric on $\alpha\beta$ and therefore it is the sum of two independent irreducible spin tensors corresponding to the completely symmetric $\yng(3)$ and corner $\yng(2,1)$ tabluex. Both of these should vanish separately. Taking $\mathcal{J}_{(\alpha\beta\gamma)} = 0$ we see that:

$$
4e + 8b + 6 = 0 \quad . \tag{C.10}
$$

Then setting $\mathcal{J}^{\gamma}_{\ \beta\gamma} = 0$ we have:

$$- 8e + 2b - 3 = 0 \quad . \tag{C.11}$$

Thus,

$$e = b = -\tfrac{1}{2} \quad . \tag{C.12}$$

We now turn to the last terms, they are proportional to the Lorentz generator. Looking at non-linear terms involving R we have:

$$\begin{aligned}
&[g - 2a]\epsilon_c{}^{fd}(\gamma_d)_{\alpha\beta}R^2\mathcal{M}_f = 0 \quad , \\
&[f - 4b]\tfrac{2}{3}(\gamma^d)_{\alpha\beta}\epsilon_{dc}{}^{f}\nabla^2 R\mathcal{M}_f = 0 \quad , \\
\rightarrow \quad & g = 2a = -1 \qquad f = 4b = -2 \quad .
\end{aligned}$$

Where we have used the following fact to extract these contributions:

$$\nabla_\alpha\nabla_\beta R = \tfrac{1}{2}(\gamma^d)_{\alpha\beta}(\nabla_d R) - C_{\alpha\beta}\nabla^2 R \quad . \tag{C.13}$$

The remaining terms in this Bianchi identity are:

$$\begin{aligned}
\Big\{&(\gamma^d)_{\alpha\beta}\epsilon_{dce}\widehat{\mathcal{R}}^{ef} + (\gamma^f)_{\alpha\beta}(\nabla_c R) + 2(\gamma_c)_{(\alpha}{}^{\delta}\nabla_{\beta)}\Sigma_\delta{}^f \\
&-\tfrac{2}{3}b(\gamma_c\gamma^f\gamma^d)_{(\alpha\beta)}(\nabla_d R) - (\gamma^d)_{\alpha\beta}(\nabla_d R)\delta_c{}^f\Big\}\mathcal{M}_f = 0 \quad .
\end{aligned} \tag{C.14}$$

After converting the free vector index into two symmetric spinor indices by contracting with $(\gamma^c)_{\gamma\delta}$ we have an expression of the form $\mathcal{J}^f_{\alpha\beta\gamma\delta}\mathcal{M}_f = 0$. This tensor is the product of two rank two symmetric spin tensors and has the following decomposition in terms of tabluex: $\ydiagram{2} \otimes \ydiagram{2} = \ydiagram{4} \oplus \ydiagram{2,2} \oplus \ydiagram{3,1}$. The completely symmetric term vanishes identically. The box diagram $\sim C^{\gamma\alpha}C^{\delta\beta}\mathcal{J}^f_{\alpha\beta\gamma\delta}\mathcal{M}_f$ takes the form:

$$0 = \{-2\nabla^c R - 12\nabla^\delta\Sigma_\delta{}^c + 8b\nabla^c R + 2\nabla^c R\}\mathcal{M}_c \quad .$$

Which implies that $\nabla^\sigma\Sigma_\sigma{}^f = -\frac{1}{3}\nabla^f R$. The gun diagram $\sim C^{\gamma\alpha}\mathcal{J}^f_{\alpha(\beta\delta)\gamma}\mathcal{M}_f$ takes the form:

$$0 = \{-4(\gamma^e)_{\beta\delta}\widehat{\mathcal{R}}_e{}^f - 8\nabla_{(\beta}\Sigma_{\delta)}{}^f + (2 + 2 - \tfrac{8}{3})\epsilon^{cde}(\gamma_e)_{\beta\delta}\nabla_d R\}\mathcal{M}_f \quad .$$

Which implies that $\nabla_{(\beta}\Sigma_{\delta)}{}^f = \frac{1}{6}\epsilon^{fde}(\gamma_e)_{\delta\beta}\nabla_d R - \frac{1}{2}(\gamma^e)_{\beta\delta}\widehat{\mathcal{R}}_e{}^f$. Thus, the spinorial derivative of $\Sigma_\alpha{}^f$ takes the form:

$$\nabla_\alpha\Sigma_\beta{}^f = -\tfrac{1}{4}(\gamma^e)_{\alpha\beta}\widehat{\mathcal{R}}_e{}^f + \tfrac{1}{6}[C_{\alpha\beta}\eta^{fd} + \tfrac{1}{2}\epsilon^{fde}(\gamma_e)_{\alpha\beta}]\nabla_d R \quad . \tag{C.15}$$

This completes the analysis of the spin-spin-vector Bianchi identity. We now move on to the spin-vector-vector Bianchi identity:

$$[[\nabla_\alpha, \nabla_b\}, \nabla_c\} + [[\nabla_b, \nabla_c\}, \nabla_\alpha\} + [[\nabla_c, \nabla_\alpha\}, \nabla_b\} = 0 \quad .$$

This identity is satisfied identically, yielding no further constraints. The final identity is all vector derivatives: $[[\nabla_{[a}, \nabla_b\}, \nabla_{c]}\} = 0$. This identity yields some more differential constraints which are of no consequence to the derivations in the body of this paper.

D Review of $Spin(7)$ Holonomy Manifolds

This appendix contains a brief review of some of the relevant aspects of $Spin(7)$ holonomy manifolds. An elegant discussion can be found in [43]. On an Riemannian manifold X of dimension n, the spin connection ω is, in general, an $SO(n)$ gauge field. If we parallel transport a spinor ψ around a closed path γ, the spinor comes back as $U\psi$, where $U = Pexp \int_\gamma \omega \, dx$ is the path ordered exponential of ω around the curve γ.

A compactification of $\mathcal{M}$-theory (or string theory) on X preserves some amount of supersymmetry if X admits one (or more) covariantly constant spinors. Such spinors return upon parallel transport to their original values, i.e. they satisfy $U\psi = \psi$. The holonomy of the manifold is then a (proper) subgroup of $SO(n)$. A $Spin(7)$ holonomy manifold is an eight-dimensional manifold, for which one such spinor exists. Therefore, if we compactify $\mathcal{M}$-theory on these manifolds we obtain an $\mathcal{N} = 1$ theory in three dimensions. $Spin(7)$ is a subgroup of $GL(8, \mathbb{R})$ defined as follows. Introduce on $\mathbb{R}^8$ the coordinates $(x_1, \ldots, x_8)$ and the four-form $dx_{ijkl} = dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. Define a self-dual 4-form Ω on $\mathbb{R}^8$ by

$$\begin{aligned} \Omega &= dx_{1234} + dx_{1256} + dx_{1278} + dx_{1357} - dx_{1368} \\ &- dx_{1458} - dx_{1467} - dx_{2358} - dx_{2367} - dx_{2457} \\ &+ dx_{2468} + dx_{3456} + dx_{3478} + dx_{5678} \,. \end{aligned} \tag{D.1}$$

The subgroup of $GL(8, \mathbb{R})$ preserving Ω is the holonomy group $Spin(7)$. It is a compact, connected, simply connected, semisimple, twenty-one-dimensional Lie group, which is isomorphic to the double cover of $SO(7)$. Many of the mathematical properties of $Spin(7)$ holonomy manifolds are discussed in detail in [43]. Let us here only mention that these manifolds are Ricci flat but are, in general, not Kähler.

The cohomology of a compact $Spin(7)$ holonomy manifold can be decomposed into the following representations of $Spin(7)$

$$\begin{aligned} H^0(X,\mathbb{R}) &= \mathbb{R} \;, \\ H^1(X,\mathbb{R}) &= 0 \;, \\ H^2(X,\mathbb{R}) &= H^2_{\mathbf{21}}(X,\mathbb{R}) \;, \\ H^3(X,\mathbb{R}) &= H^3_{\mathbf{48}}(X,\mathbb{R}) \;, \\ H^4(X,\mathbb{R}) &= H^4_{\mathbf{1}^+}(X,\mathbb{R}) \oplus H^4_{\mathbf{27}^+}(X,\mathbb{R}) \oplus H^4_{\mathbf{35}^-}(X,\mathbb{R}) \;, \\ H^5(X,\mathbb{R}) &= H^5_{\mathbf{48}}(X,\mathbb{R}) \;, \\ H^6(X,\mathbb{R}) &= H^6_{\mathbf{21}}(X,\mathbb{R}) \;, \\ H^7(X,\mathbb{R}) &= 0 \;, \\ H^8(X,\mathbb{R}) &= \mathbb{R} \;. \end{aligned} \tag{D.2}$$

where the label " $\pm$ " indicates self-dual and anti-self-dual four-forms respectively and the

subindex indicates the representation. The Cayley calibration Ω belongs to the cohomology $H^4_{\mathbf{1}^+}(X, \mathrm{I\!R})$.

Next we will briefly discuss deformations of the Cayley calibration. More details can be found in [43] and [53]. The tangent space to the family of torsion-free $Spin(7)$ structures, up to diffeomorphism is naturally isomorphic to the direct sum $H^4_{\mathbf{1}^+}(X, \mathrm{I\!R}) \oplus H^4_{\mathbf{35}^-}(X, \mathrm{I\!R})$ if X is compact and the holonomy is $Spin(7)$ and not some proper subgroup. Thus, if the holonomy is $Spin(7)$ the family has dimension $1 + b_4^-$, and the infinitesimal variations in Ω are of the form $c\Omega + \xi$ where ξ a harmonic anti-self-dual four-form and c is a number.

When we are moving in moduli space along the "radial direction" ϕ, the Cayley calibration deformation is

$$\delta\Omega = K\delta\phi\Omega \,, \tag{D.3}$$

or in other words

$$\frac{\partial\Omega}{\partial\phi} = K\Omega \,. \tag{D.4}$$

If we consider infinitesimal displacements in moduli space along the other b_4^- directions, then the Cayley calibration deformation is

$$\delta\Omega = \delta\phi^A(\xi_A - K_A\Omega) \,, \tag{D.5}$$

or in other words

$$\frac{\partial\Omega}{\partial\phi^A} = \xi_A - K_A\Omega \,, \tag{D.6}$$

where ξ_A are the anti-self-dual harmonic four-forms. If the movement in the moduli space is not along some particular direction then

$$\delta\Omega = \delta\phi^A\, \xi_A + (\delta\phi\, K - \delta\phi^A\, K_A)\Omega \,. \tag{D.7}$$

We note that the potential

$$P = \frac{1}{2}\ln(\int_{M_8} \Omega \wedge \star\Omega) \,, \tag{D.8}$$

generates $K = \partial P$ and $K_A = -\partial_A P$. The fact that

$$\int_{M_8} \Omega \wedge \star\Omega = 14\mathcal{V}_{M_8} = e^{2P} \,, \tag{D.9}$$

fixes $K = 2$, where $\mathcal{V}_{M_8}$ is the volume of the internal manifold.

E Dimensional Reduction of the Einstein-Hilbert Term

In this appendix we present the technical details related to the compactification of the Einstein-Hilbert term. We treat first the zero flux case and then we calculate the reduction for the non-zero background flux case. As usual the Greek indices refer to the external space, the small Latin indices refer to the internal space, and finally the capital Latin indices refer to the entire eleven dimensional space.

E.1 Zero Flux Case

We start with the following ansatz for the inverse metric

$$g^{mn}(x,y) = \hat{g}^{mn}(y) + \phi(x)\hat{g}^{mn}(y) + \sum_{A=1}^{b_4^-} \phi^A(x)\; h_A^{mn}(x,y) + \dots\,, \tag{E.1.1}$$

where we have denoted by $g^{mn}(x,y)$ the inverse metric of $g_{mn}(x,y)$ and by $\hat{g}^{mn}(x)$ the inverse metric of $\hat{g}_{mn}(x)$

$$g_{mn}(x,y)\, g^{np}(x,y) = \delta_m{}^p \quad , \quad \hat{g}_{mn}(y)\,\hat{g}^{np}(y) = \delta_m{}^p\,. \tag{E.1.2}$$

Due to these facts we obtain that

$$h_A^{mn}(x,y) = -\hat{g}^{ma}(y)\, e_{A\,ab}(y)\,\hat{g}^{bn}(y)\,. \tag{E.1.3}$$

The tracelessness of $e_{A\,ab}$ implies the tracelessness of h_A^{mn}. The ansatz (3.1.9) implies that the only non-zero Christoffel symbols are

$$\begin{aligned}
\Gamma^\alpha_{\mu\nu} &= \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right),\\
\Gamma^\alpha_{m\nu} &= 0\,,\\
\Gamma^\alpha_{mn} &= -\frac{1}{2} g^{\alpha\beta}\left(\partial_\beta g_{mn}\right),\\
\Gamma^a_{mn} &= \frac{1}{2} g^{ab}\left(\partial_m g_{bn} + \partial_n g_{mb} - \partial_b g_{mn}\right),\\
\Gamma^a_{\mu n} &= \frac{1}{2} g^{ab}\left(\partial_\mu g_{bn}\right),\\
\Gamma^a_{\mu\nu} &= 0\,.
\end{aligned} \tag{E.1.4}$$

Using the following definition of the Ricci tensor

$$R^{(11)}_{MN} = \partial_A \Gamma^A_{MN} - \partial_N \Gamma^A_{MA} + \Gamma^A_{MN}\Gamma^B_{AB} - \Gamma^A_{MB}\Gamma^B_{AN}\,, \tag{E.1.5}$$

we can derive the expression for the eleven-dimensional Ricci scalar

$$\begin{aligned}
R^{(11)} &= R^{(3)} + R^{(8)} + g^{mn}\partial_\alpha \Gamma^\alpha_{mn} - g^{\mu\nu}\partial_\nu \Gamma^a_{\mu a} + g^{\mu\nu}\Gamma^\alpha_{\mu\nu}\Gamma^b_{\alpha b} + g^{mn}\Gamma^\alpha_{mn}\Gamma^\beta_{\alpha\beta}\\
&+ g^{mn}\Gamma^\alpha_{mn}\Gamma^b_{\alpha b} - \left[g^{\mu\nu}\Gamma^a_{\mu b}\Gamma^b_{a\nu} + g^{mn}\Gamma^a_{m\beta}\Gamma^\beta_{an} + g^{mn}\Gamma^\alpha_{mb}\Gamma^b_{\alpha n}\right].
\end{aligned} \tag{E.1.6}$$

where $R^{(3)}$ is the three-dimensional Ricci scalar and $R^{(8)}$ is the eight-dimensional Ricci scalar. We can determine that

$$\begin{aligned}
\int_{\mathcal{M}} d^{11}x\,\sqrt{-g^{(11)}}\,R^{(11)} &= \int_{\mathcal{M}} d^{11}x\,\sqrt{-g^{(11)}}\,\Big\{R^{(3)} + g^{\mu\nu}\Gamma^a_{\mu a}\Gamma^b_{\nu b} - (\partial_\alpha g^{mn})\Gamma^\alpha_{mn}\\
&- \left[g^{\mu\nu}\Gamma^a_{\mu b}\Gamma^b_{\nu a} + g^{mn}\Gamma^a_{\beta m}\Gamma^\beta_{an} + g^{mn}\Gamma^\alpha_{mb}\Gamma^b_{\alpha n}\right]\Big\}\,,
\end{aligned} \tag{E.1.7}$$

where we have integrated by parts with respect to the internal coordinates and we have used the fact that the internal manifold is Ricci flat, i.e. $R^{(8)} = 0$. It is easy to see that

$$\int_{\mathcal{M}} d^{11}x\sqrt{-g^{(11)}}\, g^{\mu\nu}\,\Gamma^a_{\mu a}\,\Gamma^b_{\nu b} = \mathcal{V}_{M_8}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{16\,(\partial_\alpha\phi)\,(\partial^\alpha\phi)\Big\}\,, \tag{E.1.8}$$

$$\int_{\mathcal{M}} d^{11}x\sqrt{-g^{(11)}}\,(\partial_\alpha g^{mn})\Gamma^{\alpha}_{mn} = 2\,\mathcal{V}_{M_8}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{2\,(\partial_\alpha\phi)\,(\partial^\alpha\phi)$$

$$+\sum_{A,B=1}^{b_4^-}\mathcal{L}_{AB}\,(\partial_\alpha\phi^A)\,(\partial^\alpha\phi^B)\Big\}\,, \tag{E.1.9}$$

$$\begin{aligned}&\int_{\mathcal{M}} d^{11}x\sqrt{-g^{(11)}}\,\left[g^{\mu\nu}\Gamma^{a}_{\mu b}\Gamma^{b}_{\nu a}+g^{mn}\Gamma^{a}_{\beta m}\Gamma^{\beta}_{an}+g^{mn}\Gamma^{\alpha}_{mb}\Gamma^{b}_{\alpha n}\right]\\ &= -\,\mathcal{V}_{M_8}\int_{M_3} d^3x\sqrt{-g^{(3)}}\,\Big\{2\,(\partial_\alpha\phi)\,(\partial^\alpha\phi)\\ &+\sum_{A,B=1}^{b_4^-}\mathcal{L}_{AB}(\partial_\alpha\phi^A)\,(\partial^\alpha\phi^B)\Big\}\,,\end{aligned} \tag{E.1.10}$$

where $\mathcal{L}_{AB}$ was defined in (3.1.21) and $\mathcal{V}_{M_8}$ represents the volume of the internal manifold and it is defined in (3.1.19).

We know that after compactification we arrive in the string frame even if we started in eleven dimensions in the Einstein frame. Therefore we have to perform a Weyl transformation for the external metric. The fact that we do not see any exponential of the radial modulus seating in front of $R^{(3)}$ is because we have consistently neglected higher order contributions in moduli fields. However it is not difficult to realize that the Weyl transformation that has to be performed is

$$g_{\alpha\beta}\to e^{-8\phi}\,g_{\alpha\beta}\,. \tag{E.1.11}$$

The only visible change in this order of approximation is the coefficient in front of the kinetic term for radion. All the other terms in the action remain unchanged. Therefore the Einstein-Hilbert term is

$$\begin{aligned}\frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}} d^{11}x\,\sqrt{-g^{(11)}}\,R^{(11)} &= \frac{1}{2\kappa_3^2}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{R^{(3)}-18(\partial_\alpha\phi)\,(\partial^\alpha\phi)\\ &-\sum_{A,B=1}^{b_4^-}\mathcal{L}_{AB}\,(\partial_\alpha\phi^A)\,(\partial^\alpha\phi^B)\Big\}+\ldots\,.\end{aligned} \tag{E.1.12}$$

E.2 Non-Zero Flux Case

It is easy to derive an expression for the Ricci scalar in the non-zero background case. For this task we rewrite the warped metric (3.2.1) as

$$\widetilde{g}_{MN} = \Omega^2(y)\,\bar{g}_{MN}\,, \tag{E.2.1}$$

where $\Omega(y) = e^{\Delta(y)/3}$ and therefore

$$\bar{g}^{(11)}_{MN}(x,y) = \begin{pmatrix} g^{(3)}_{\mu\nu}(x) & 0 \\ 0 & \Omega^{-3}(y)\,g^{(8)}_{mn}(x,y)\end{pmatrix}\,. \tag{E.2.2}$$

The Christoffel symbols that correspond to the metric (E.2.2) are

$$
\begin{aligned}
\bar{\Gamma}^{\alpha}_{\mu\nu} &= \Gamma^{\alpha}_{\mu\nu}\,, \\
\bar{\Gamma}^{\alpha}_{m\nu} &= \Gamma^{\alpha}_{m\nu}\,, \\
\bar{\Gamma}^{\alpha}_{mn} &= \Omega^{-3}\,\Gamma^{\alpha}_{mn}\,, \\
\bar{\Gamma}^{a}_{mn} &= \Gamma^{a}_{mn} - \frac{3}{2}\left[\delta^a_m\partial_n + \delta^a_n\partial_m - g_{mn}g^{ab}\partial_b\right]\ln\Omega\,, \\
\bar{\Gamma}^{a}_{\mu n} &= \Gamma^{a}_{\mu n}\,, \\
\bar{\Gamma}^{a}_{\mu\nu} &= \Gamma^{a}_{\mu\nu}\,,
\end{aligned}
\tag{E.2.3}
$$

where the unbarred symbols are computed in (E.1.4). We can repeat the computation for the Ricci scalar that correspond to the metric (E.2.2) and at the end we will obtain a similar formula to (E.1.6). Due to the simple relations (E.2.3) between the Christoffel symbols, the Ricci scalar for the metric (E.2.2) reduces to

$$
\bar{R}^{(11)} = R^{(11)} + 21\,\Omega^3\left[g^{ab}\nabla_a\nabla_b\ln\Omega - \frac{9}{2}\,g^{ab}\,\nabla_a\ln\Omega\,\nabla_b\ln\Omega\right]\,, \tag{E.2.4}
$$

where $R^{(11)}$ is given in (E.1.6).

To compute the Ricci scalar that corresponds to the metric (3.2.1) we have to perform the conformal transformation (E.2.1). The result expressed in terms of the warp factor $\Delta(y)$ is

$$
\widetilde{R}^{(11)} = e^{-2\Delta(y)/3}R^{(11)} + e^{\Delta(y)/3}\,\left[\frac{1}{3}\,g^{ab}\nabla_a\nabla_b\Delta(y) - \frac{1}{2}\,g^{ab}\,\nabla_a\Delta(y)\,\nabla_b\Delta(y)\right]\,. \tag{E.2.5}
$$

Using the fact that the second term in (E.2.5) produces a total derivative term which vanishes by Stokes' theorem and the last term is subleading, we obtain that

$$
\int_{\mathcal{M}} d^{11}x\,\sqrt{-\widetilde{g}^{(11)}}\,\widetilde{R}^{(11)} \;=\; \int_{\mathcal{M}} d^{11}x\,\sqrt{-g}\,R^{(11)}\,e^{-\Delta(y)} + \dots\,. \tag{E.2.6}
$$

As expected, to leading order the kinetic coefficients receive no corrections from warping. Therefore we conclude that

$$
\begin{aligned}
\frac{1}{2\kappa_{11}^2}\int_{\mathcal{M}} d^{11}x\,\sqrt{-\widetilde{g}^{(11)}}\,\widetilde{R}^{(11)} \;&=\; \frac{1}{2\kappa_3^2}\int_{M_3} d^3x\,\sqrt{-g^{(3)}}\,\Big\{R^{(3)} - 18(\partial_\alpha\phi)\,(\partial^\alpha\phi) \\
&\quad - \sum_{A,B=1}^{b_4^-}\mathcal{L}_{AB}\,(\partial_\alpha\phi^A)\,(\partial^\alpha\phi^B)\Big\} + \dots\,.
\end{aligned}
\tag{E.2.7}
$$

References

[1] M. Rocek and P. van Nieuwenhuizen. *"$N \geq 2$ Supersymmetric Chern-Simons Terms as $D = 3$ Extended Conformal Supergravity"*. *Class. Quant. Grav.*, **3**:43, (1986).

[2] P. van Nieuwenhuizen. *"$D = 3$ Conformal Supergravity and Chern-Simons Terms"*. *Phys. Rev.*, **D32**:872, (1985).

[3] A. Karlhede, U. Lindstrom, M. Rocek, and P. van Nieuwenhuizen. *"On 3-D Nonlinear Vector-Vector Duality"*. *Phys. Lett.*, **B186**:96, (1987).

[4] A. Karlhede, U. Lindstrom, M. Rocek, and P. van Nieuwenhuizen. *"Supersymmetric Vector-Vector Duality"*. *Class. Quant. Grav.*, **4**:549, (1987).

[5] J. H. Park. *"Superconformal Symmetry in Three-dimensions"*. *J. Math. Phys.*, **41**:7129, (2000), hep-th/9910199.

[6] E. Ivanov, S. Krivonos, and O. Lechtenfeld. *"Double-vector Multiplet and Partially Broken $N = 4$, $D = 3$ Supersymmetry"*. *Phys. Lett.*, **B487**:192, (2000), hep-th/0006017.

[7] B. M. Zupnik. *"Partial Spontaneous Breakdown of Three-dimensional $N = 2$ Supersymmetry"*. *Theor. Math. Phys.*, **123**:463, (2000), hep-th/9905108.

[8] B. M. Zupnik. *"Harmonic Superspaces for Three-Dimensional Theories"*. (1997), hep-th/9804167.

[9] J Alexandre. *"From $\mathcal{N} = 1$ to $\mathcal{N} = 2$ Supersymmetries in $2 + 1$ Dimensions"*. *Phys. Rev.*, **D67**:105017, (2003), hep-th/0301034.

[10] B. de Wit, A. K. Tollsten, and H. Nicolai. *"Locally Supersymmetric $D = 3$ Nonlinear Sigma Models"*. *Nucl. Phys.*, **B392**:3, (1993), hep-th/9208074.

[11] B. de Wit, I. Herger, and H. Samtleben. *"Gauged Locally Supersymmetric $D = 3$ Nonlinear Sigma Models"*. *Nucl. Phys.*, **B671**:175–216, (2003), hep-th/0307006.

[12] M. Brown and S. J. Gates, Jr. *"Superspace Bianchi Identities and the Supercovariant Derivative"*. *Ann. Phys.*, **122**:443, (1979).

[13] A. Strominger. *"Superstrings with Torsion"*. *Nucl. Phys.*, **B274**:253, (1986).

[14] B. de Wit, D. J. Smit, and N. D. Hari Dass. *"Residual Supersymmetry of Compactified $D = 10$ Supergravity"*. *Nucl. Phys.*, **B283**:165, (1987).

[15] K. Becker and M. Becker. *"$\mathcal{M}$-theory on Eight-Manifolds"*. *Nucl. Phys.*, **B477**:155, (1996), hep-th/9605053.

[16] S. Sethi, C. Vafa, and E. Witten. *"Constraints on Low-Dimensional String Compactifications"*. *Nucl. Phys.*, **B480**:213, (1996), hep-th/9606122.

[17] K. Dasgupta, G. Rajesh, and S. Sethi. *"$\mathcal{M}$ Theory, Orientifolds and G-flux"*. *JHEP*, **08**:023, (1999), hep-th/9908088.

[18] T. R. Taylor and C. Vafa. *"RR Flux on Calabi-Yau and Partial Supersymmetry Breaking"*. *Phys. Lett.*, **B474**:130, (2000), hep-th/9912152.

[19] S. Gukov, C. Vafa, and E. Witten. *"CFT's from Calabi-Yau Four-folds"*. *Nucl. Phys.*, **B584**:69, (2000), hep-th/9906070.

[20] M. Haack and J. Louis. *"$\mathcal{M}$-theory Compactified on Calabi-Yau Fourfolds with Background Flux"*. *Phys. Lett.*, **B507**:296, (2001), hep-th/0103068.

[21] S. B. Giddings, S. Kachru, and J. Polchinski. *"Hierarchies from Fluxes in String Compactifications"*. *Phys. Rev.*, **D66**:106006, (2002), hep-th/0105097.

[22] M. Berg, M. Haack, and H. Samtleben. *"Calabi-Yau Fourfolds with Flux and Supersymmetry Breaking"*. *JHEP*, **04**:046, (2003), hep-th/0212255.

[23] E. Witten. *"Is Supersymmetry Really Broken?"*. *Int. J. Mod. Phys.*, **A10**:1247, (1995), hep-th/9409111.

[24] E. Witten. *"Strong Coupling and the Cosmological Constant"*. *Mod. Phys. Lett.*, **A10**:2153, (1995), hep-th/9506101.

[25] M. Grana and J. Polchinski. *"Supersymmetric Three-Form Flux Perturbations on AdS_5"*. *Phys. Rev.*, **D63**:026001, (2001), hep-th/0009211.

[26] S. S. Gubser. *"Supersymmetry and F-theory Realization of the Deformed Conifold with Three-Form Flux"*. (2000), hep-th/0010010.

[27] I. R. Klebanov and M. J. Strassler. *"Supergravity and a Confining Gauge Theory: Duality Cascades and χSB-Resolution of Naked Singularities"*. *JHEP*, **08**:052, (2000), hep-th/0007191.

[28] J. M. Maldacena. *"The Large N Limit of Superconformal Field Theories and Supergravity"*. *Adv. Theor. Math. Phys.*, **2**:231, (1998), hep-th/9711200.

[29] I. R. Klebanov and E. Witten. *"Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity"*. *Nucl. Phys.*, **B536**:199, (1998), hep-th/9807080.

[30] I. R. Klebanov and A. A. Tseytlin. *"Gravity Duals of Supersymmetric $SU(N)\times SU(N+M)$ Gauge Theories"*. *Nucl. Phys.*, **B578**:123, (2000), hep-th/0002159.

[31] J. Polchinski and M. J. Strassler. *"The String Dual of a Confining Four-Dimensional Gauge Theory"*. (2000), hep-th/0003136.

[32] S. L. Shatashvili and C. Vafa. *"Superstrings and Manifold of Exceptional Holonomy"*. (1994), hep-th/9407025.

[33] G. Papadopoulos and P. K. Townsend. *"Compactification of $D = 11$ Supergravity on Spaces of Exceptional Holonomy"*. *Phys. Lett.*, **B357**:300, (1995), hep-th/9506150.

[34] K. Becker. *"A Note on Compactifications on Spin(7)-Holonomy Manifolds"*. *JHEP*, **05**:003, (2001), hep-th/0011114.

[35] B. Acharya, X. de la Ossa, and S. Gukov. *"G-Flux, Supersymmetry and Spin(7) Manifolds"*. *JHEP*, **09**:047, (2002), hep-th/0201227.

[36] M. Becker and D. Constantin. *"A Note on Flux Induced Superpotentials in String Theory"*. *JHEP*, **08**:015, (2003), hep-th/0210131.

[37] S. Gukov. *"Solitons, Superpotentials and Calibrations"*. *Nucl. Phys.*, **B574**:169, (2000), hep-th/9911011.

[38] S. Gukov and J. Sparks. *"$\mathcal{M}$-Theory on Spin(7) Manifolds. I"*. *Nucl. Phys.*, **B625**:3, (2002), hep-th/0109025.

[39] M. Cvetic, G. W. Gibbons, H. Lu, and C. N. Pope. *"New Complete Non-Compact Spin(7) Manifolds"*. *Nucl. Phys.*, **B620**:29, (2002), hep-th/0103155.

[40] M. T. Grisaru, M. E. Knutt-Wehlau, and W. Siegel. *"A Superspace Normal Coordinate Derivation of the Density Formula"*. *Nucl. Phys.*, **B523**:663, (1998), hep-th/9711120.

[41] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt-Wehlau, and W. Siegel. *"Component Actions from Curved Superspace: Normal Coordinates and Ectoplasm"*. *Phys. Lett.*, **B421**:203, (1998), hep-th/9711151.

[42] S. J. Gates, Jr. *"Ectoplasm has no Topology"*. *Nucl. Phys.*, **B541**:615, (1999), hep-th/9809056.

[43] D. D. Joyce. "Compact Manifolds with Special Holonomy". Oxford Science Publications, Oxford, UK, (2000).

[44] E. Cremmer, B. Julia, and J. Scherk. *"Supergravity Theory in 11 Dimensions"*. *Phys. Lett.*, **B76**:409, (1978).

[45] G. W. Gibbons, D. N. Page, and C. N. Pope. *"Einstein Metrics on S^3, R^3 and R^4 Bundels"*. *Commun. Math. Phys.*, **127**:529, (1990).

[46] T. Banks and M. B. Green. *"Non-Perturbative Effects in $AdS_5 \times S^5$ String Theory and $d=4$ SUSY Yang-Mills"*. *JHEP*, **05**:002, (1998), hep-th/9804170.

[47] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin. *"Coupling Constant Dependence in the Thermodynamics of $N=4$ Supersymmetric Yang-Mills Theory"*. *Nucl. Phys.*, **B534**:202, (1998), hep-th/9805156.

[48] D. Constantin. *"Work in Progress"*.

[49] S. Kachru, R. Kallosh, A. Linde, and S. P. Trivedi. *"De Sitter Vacua in String Theory"*. *Phys. Rev.*, **D68**:046005, (2003), hep-th/0301240.

[50] D. Martelli and J. Sparks. *"G-Structures, Fluxes and Calibrations in $\mathcal{M}$-Theory"*. *Phys. Rev.*, **D68**:085014, (2003), hep-th/0306225.

[51] A. R. Frey and M. Grana. *"Type IIB Solutions with Interpolating Supersymmetries"*. (2003), hep-th/0307142.

[52] K. Becker, M. Becker, and R. Sriharsha. *"PP-waves, $\mathcal{M}$-Theory and Fluxes"*. (2003), hep-th/0308014.

[53] Spiro Karigiannis. *"Deformations of G_2 and $Spin(7)$ Structures on Manifolds"*. *Ph.D. Thesis - Harvard University*, (2003), math.DG/0301218.